THE ALGEBRA AND MODEL THEORY OF TAME VALUED FIELDS

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Abstract. A henselian valued field K is called a tame field if its algebraic closure $\tilde{K}$ is a tame extension, that is, the ramification field of the normal extension $\tilde{K}|K$ is algebraically closed. Every algebraically maximal Kaplansky field is a tame field, but not conversely. We develop the algebraic theory of tame fields and then prove Ax–Kochen–Ershov Principles for tame fields. This leads to model completeness and completeness results relative to value group and residue field. As the maximal immediate extensions of tame fields will in general not be unique, the proofs have to use much deeper valuation theoretical results than those for other classes of valued fields which have already been shown to satisfy Ax–Kochen–Ershov Principles. The results of this paper have been applied to gain insight in the Zariski space of places of an algebraic function field, and in the model theory of large fields.

1. Introduction

In this paper, we consider valued fields. By (K,v) we mean a field K equipped with a valuation v. We denote the value group by vK, the residue field by Kv and the valuation ring by $\mathcal{O}_v$ or $\mathcal{O}_K$. For elements $a\in K$, the value is denoted by va, and the residue by av. For a polynomial $f\in K[X]$, we denote by fv the polynomial in $Kv[X]$ that is obtained from f by replacing all its coefficients by their residues.

We write a valuation in the classical additive (Krull) way, that is, the value group is an additively written ordered abelian group, the homomorphism property of v says that $vab=va+vb$, and the ultrametric triangle law says that $v(a+b)\geq\min\{va,vb\}$. Further, we have the rule $va=\infty\Leftrightarrow a=0$. We take $\mathcal{L}_{\mathrm{VF}}=\{+,-,\cdot,\ ^{-1},0,1,\mathcal{O}\}$ to be the language of valued fields, where $\mathcal{O}$ is a binary relation symbol for valuation divisibility. That is, $\mathcal{O}(a,b)$ will be interpreted by $va\geq vb$, or equivalently, a/b being an element of the valuation ring $\mathcal{O}_v$. We will write $\mathcal{O}(X)$ in place of $\mathcal{O}(X,1)$ (note that $\mathcal{O}(a,1)$ says that $va\geq v1=0$, i.e., $a\in\mathcal{O}_v$).

When we talk of a valued field extension $(L|K,v)$ we mean that (L,v) is a valued field, $L|K$ a field extension, and K is endowed with the restriction of v.

A valued field is **henselian** if it satisfies Hensel's Lemma, or equivalently, if it admits a unique extension of the valuation to every algebraic extension field; see [Ri], [W], [E–P].

For (K,v) and (L,v) to be elementarily equivalent in the language of valued fields, it is necessary that vK and vL are elementarily equivalent in the language $\mathcal{L}_{\mathrm{OG}}$ $=\{+,-,0,<\}$ of ordered groups, and that Kv and Lv are elementarily equivalent in

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the language $\mathcal{L}_{\mathrm{F}} = \{+,-,\cdot,\ ^{-1},0,1\}$ of fields (or in the language $\mathcal{L}_{\mathrm{R}} = \{+,-,\cdot,0,1\}$ of rings). This is because elementary sentences about the value group and about the residue field can be encoded in the valued field itself.

Our main concern in this paper is to find additional conditions on (K,v) and (L,v) under which these necessary conditions are also sufficient, i.e., the following **Ax–Kochen–Ershov Principle** (in short: AKE$^{\equiv}$ Principle) holds:

$$vK \equiv vL \;\wedge\; Kv \equiv Lv \quad\Longrightarrow\quad (K,v) \equiv (L,v)\;. \tag{1}$$

An AKE$^{\prec}$ Principle is the following analogue for elementary extensions:

$$(K,v) \subseteq (L,v) \;\wedge\; vK \prec vL \;\wedge\; Kv \prec Lv \quad\Longrightarrow\quad (K,v) \prec (L,v)\;. \tag{2}$$

If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{M}'$ a substructure of $\mathcal{M}$, then we will say that $\mathcal{M}'$ is existentially closed in $\mathcal{M}$ and write $\mathcal{M}' \prec_\exists \mathcal{M}$ if every existential $\mathcal{L}$-sentence with parameters from $\mathcal{M}'$ which holds in $\mathcal{M}$ also holds in $\mathcal{M}'$. For the meaning of "existentially closed in" in the setting of valued fields and of ordered abelian groups, see [K–P]. Inspired by Robinson's Test, our basic approach will be to ask for criteria for a valued field to be existentially closed in a given extension field. Replacing $\prec$ by $\prec_\exists$, we thus look for conditions which ensure that the following AKE$^{\exists}$ Principle holds:

$$(K,v) \subseteq (L,v) \;\wedge\; vK \prec_\exists vL \;\wedge\; Kv \prec_\exists Lv \quad\Longrightarrow\quad (K,v) \prec_\exists (L,v)\;. \tag{3}$$

The conditions

$$vK \prec_\exists vL \quad\text{and}\quad Kv \prec_\exists Lv \tag{4}$$

will be called **side conditions**. It is an easy exercise in model theoretic algebra to show that these conditions imply that vL/vK is torsion free and that $Lv|Kv$ is **regular**, i.e., the algebraic closure of Kv is linearly disjoint from Lv over Kv, or equivalently, Kv is relatively algebraically closed in Lv and $Lv|Kv$ is separable; cf. Lemma 5.3.

A valued field for which (3) holds will be called an **AKE$^{\exists}$-field**. A class **C** of valued fields will be called **AKE$^{\equiv}$-class** (or **AKE$^{\prec}$-class**) if (1) (or (2), respectively) holds for all $(K,v),(L,v) \in \mathbf{C}$, and it will be called **AKE$^{\exists}$-class** if (3) holds for all $(K,v) \in \mathbf{C}$. We will also say that **C** is **relatively complete** if it is an AKE$^{\equiv}$-class, and that **C** is **relatively model complete** if it is an AKE$^{\prec}$-class. Here, "relatively" means "relative to the value groups and residue fields".

The following elementary classes of valued fields are known to satisfy all or some of the above AKE Principles:

a) Algebraically closed valued fields satisfy all three AKE Principles. They even admit quantifier elimination; this has been shown by Abraham Robinson, cf. [Ro].

b) Henselian fields of residue characteristic 0 satisfy all three AKE Principles. These facts have been (explicitly or implicitly) shown by Ax and Kochen [AK] and independently by Ershov [Er3]. They admit quantifier elimination relative to their value group and residue field, cf. [D].

c) p-adically closed fields satisfy all three AKE Principles. Again, these fields were treated by Ax and Kochen [AK] and independently by Ershov [Er3].

d) $\wp$-adically closed fields (i.e., finite extensions of p-adically closed fields): for definitions and results see the monograph by Prestel and Roquette [P–R].

e) Finitely ramified fields: this case is a generalization of c) and d). These fields were treated by Ziegler [Zi] and independently by Ershov [Er5].

f) Algebraically maximal Kaplansky fields (see below for definitions). Again, these fields were treated by Ziegler [Zi] and independently by Ershov [Er4].

An extension $(L|K,v)$ of valued fields is called **immediate** if the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto. A valued field is called **algebraically maximal** if it does not admit proper immediate algebraic extensions; it is called **separable-algebraically maximal** if it does not admit proper immediate separable-algebraic extensions.

The **henselization** of a valued field (L,v) will be denoted by $(L,v)^h$ or simply L^h. It is "the minimal" extension of (L,v) which is henselian; for details, see Section 2. The henselization is an immediate separable-algebraic extension. Hence every separable-algebraically maximal valued field is henselian.

Every valued field admits a maximal immediate algebraic extension and a maximal immediate extension. All of the above mentioned valued fields have the common property that these extensions are unique up to valuation preserving isomorphism. This has always been a nice tool in the proofs of the model theoretic results. However, as we will show in this paper, this uniqueness is not indispensible. In its absence, one just has to work much harder.

We will show that tame valued fields (in short, "tame fields") form an AKE$^\exists$-class, and we will prove further model theoretic results for tame fields and separably tame fields. Take a henselian field (K,v), and let p denote the **characteristic exponent** of its residue field Kv, i.e., $p = \mathrm{char}\, Kv$ if this is positive, and $p = 1$ otherwise. An algebraic extension $(L|K,v)$ of a henselian field (K,v) is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $(vE : vK)$ is prime to p,
(TE2) The residue field extension $Ev|Kv$ is separable,
(TE3) The extension $(E|K,v)$ is **defectless**, i.e., $[E:K] = (vE:vK)[Ev:Kv]$.

Remark 1.1. This notion of "tame extension" does not coincide with the notion of "tamely ramified extension" as defined on page 180 of O. Endler's book [En]. The latter definition requires (TE1) and (TE2), but not (TE3). Our tame extensions are the defectless tamely ramified extensions in the sense of Endler's book. In particular, in our terminology, proper immediate algebraic extensions of henselian fields are not called tame (in fact, they cause a lot of problems in the model theory of valued fields).

A **tame field** is a henselian field for which all algebraic extensions are tame. Likewise, a **separably tame field** is a henselian field for which all separable-algebraic extensions are tame. The algebraic properties of tame fields will be studied in Section 3.1, and those of separably tame fields in Section 3.2.

If $\mathrm{char}\, Kv = 0$, then conditions (TE1) and (TE2) are void, and moreover, every finite extension of (K,v) is defectless (cf. Corollary 2.5 below). Hence every henselian valued field of residue characteristic 0 is tame.

Take a valued field (K, v) and denote the characteristic exponent of Kv by p. Then (K, v) is a **Kaplansky field** if vK is p-divisible and Kv does not admit any finite extension whose degree is divisible by p. All algebraically maximal Kaplansky fields are tame fields (cf. Corollary 3.11 below). But the converse does not hold since for a tame field it is admissible that its residue field has finite separable extensions with degree divisible by p. It is because of this fact that the uniqueness of maximal immediate extensions will in general fail (cf. [K–P–R]). This is what makes the proof of AKE Principles for tame fields so much harder than that for algebraically maximal Kaplansky fields.

In many applications (such as the proof of a Nullstellensatz), only existential sentences play a role. In these cases, it suffices to have an AKE$^\exists$ Principle at hand. There are situations where we cannot even expect more than this principle. In order to present one, we will need some definitions that will be fundamental for this paper.

Every finite extension $(L|K, v)$ of valued fields satisfies the **fundamental inequality** (cf. [En], [Ri], or [Z–S]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{5}$$

where $n = [L : K]$ is the degree of the extension, $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions of v from K to L, $\mathrm{e}_i = (v_i L : vK)$ are the respective ramification indices and $\mathrm{f}_i = [Lv_i : Kv]$ are the respective inertia degrees. The extension is called **defectless** if equality holds in (5). Note that $\mathrm{g} = 1$ if (K, v) is henselian, so the definition given in axiom (TE3) is a special case of this definition.

A valued field (K, v) is called **defectless** (or **stable**) if each of its finite extensions is defectless, and **separably defectless** if each of its finite separable extensions is defectless. If $\mathrm{char}\, Kv = 0$, then (K, v) is defectless (this is a consequence of the "Lemma of Ostrowski", cf. (10) below.

Now let $(L|K, v)$ be any extension of valued fields. Assume that $L|K$ has finite transcendence degree. Then (by Corollary 2.3 below):

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, Lv|Kv \,+\, \dim_{\mathbb{Q}} \mathbb{Q} \otimes vL/vK\;. \tag{6}$$

We will say that $(L|K, v)$ is **without transcendence defect** if equality holds in (6). If $L|K$ does not have finite transcendence degree, then we will say that $(L|K, v)$ is without transcendence defect if every subextension of finite transcendence degree is. In Section 5.2 we will prove:

Theorem 1.2. *Every extension without transcendence defect of a henselian defectless field satisfies the AKE$^\exists$ Principle.*

Note that it is not in general true that an extension of henselian defectless fields will satisfy the AKE$^\prec$ Principle. There are extensions without transcendence defect that satisfy the side conditions, for which the lower field is algebraically closed (or just henselian) while the upper field is not. A different and particularly interesting example is given in Theorem 3 of [K5].

A valued field (K, v) has **equal characteristic** if $\mathrm{char}\, K = \mathrm{char}\, Kv$. The following is the main theorem of this paper:

Theorem 1.3. *The class of all tame fields is an $AKE^{\exists}$-class and an $AKE^{\prec}$-class. The class of all tame fields of equal characteristic is an $AKE^{\equiv}$-class.*

This theorem, originally proved in [K1], has been applied in [K6] to study the structure of the Zariski space of all places of an algebraic function field in positive characteristic.

As an immediate consequence of the foregoing theorem, we get the following criterion for decidability:

Theorem 1.4. *Let (K, v) be a tame field of equal characteristic. Assume that the theories* $\mathrm{Th}(vK)$ *of its value group (as an ordered group) and* $\mathrm{Th}(Kv)$ *of its residue field (as a field) both admit recursive elementary axiomatizations. Then also the theory of (K, v) as a valued field admits a recursive elementary axiomatization and is decidable.*

Indeed, the axiomatization of $\mathrm{Th}(K, v)$ can be taken to consist of the axioms of tame fields of equal characteristic $\mathrm{char}\, K$, together with the translations of the axioms of $\mathrm{Th}(vK)$ and $\mathrm{Th}(Kv)$ to the language of valued fields (cf. Lemma 4.1).

As an application, we will prove Theorem 7.7 in Section 7.1 which includes the following decidability result:

Theorem 1.5. *Take $q = p^n$ for some prime p and some $n \in \mathbb{N}$, and an ordered abelian group Γ. Assume that Γ is divisible or elementarily equivalent to the p-divisible hull of $\mathbb{Z}$. Then the elementary theory of the power series field $\mathbb{F}_q((t^\Gamma))$ with coefficients in $\mathbb{F}_q$ and exponents in Γ, endowed with its canonical valuation v_t, is decidable.*

Here are our results for separably defectless and separably tame fields, which we will prove in Section 7.2:

Theorem 1.6. *a) Take an extension $(L|K, v)$ without transcendence defect of a henselian separably defectless field such that vK is cofinal in vL. Then the extension satisfies the $AKE^{\exists}$ Principle.*
b) Every separable extension $(L|K, v)$ of a separably tame field satisfies the $AKE^{\exists}$ Principle.

In a subsequent paper, we will discuss quantifier elimination for tame fields in a natural extension of the language of valued fields. The amc-structures described in [K3] alone are not strong enough as they do not contain sufficient information about algebraic extensions that are not tame. Predicates have to be added to the language in order to complement the amc-structures.

We will deduce our model theoretic results from two main theorems which we originally proved in [K1]. The first theorem is a generalization of the "Grauert–Remmert Stability Theorem". It deals with function fields $F|K$, i.e., F is a finitely generated field extension of K (for our purposes it is not necessary to ask that the transcendence degree is ≥ 1). For the following theorem, see [K10]:

Theorem 1.7. *Let $(F|K, v)$ be a valued function field without transcendence defect. If (K, v) is a defectless field, then also (F, v) is a defectless field.*

In [K-K1] we use Theorem 1.7 to prove:

Theorem 1.8. *Take a defectless field (K,v) and a valued function field $(F|K,v)$ without transcendence defect. Assume that $Fv|Kv$ is a separable extension and vF/vK is torsion free. Then $(F|K,v)$ admits elimination of ramification in the following sense: there is a transcendence basis $\mathcal{T}=\{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ of $(F|K,v)$ such that*

a) $vF=vK(\mathcal{T})=vK\oplus\mathbb{Z}vx_1\oplus\ldots\oplus\mathbb{Z}vx_r$,

b) $y_1v,\ldots,y_sv$ *form a separating transcendence basis of* $Fv|Kv$.

For each such transcendence basis $\mathcal{T}$ and every extension of v to the algebraic closure of F, $(F^h|K(\mathcal{T})^h,v)$ is defectless and satisfies:

1) $vF^h=vK(\mathcal{T})^h$,

2) $F^hv|K(\mathcal{T})^hv$ *is a finite separable extension and*

$$[F^h:K(\mathcal{T})^h]\;=\;[F^hv:K(\mathcal{T})^hv]\;.$$

The second fundamental theorem, originally proved in [K1], is a structure theorem for immediate function fields over tame or separably tame fields (cf. [K2], [K11]).

Theorem 1.9. *Take an immediate function field $(F|K,v)$ of transcendence degree 1. Assume that (K,v) is a tame field, or that (K,v) is a separably tame field and $F|K$ is separable. Then*

$$\text{there is } x\in F \text{ such that } (F^h,v)\;=\;(K(x)^h,v)\;. \tag{7}$$

For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless (in fact, every $x\in F\setminus K$ will then do the job). In contrast to this, the case of positive residue characteristic requires a much deeper structure theory of immediate extensions of valued fields, in order to find suitable elements x.

Theorem **??** is also used in [K-K2]. For a survey on a valuation theoretical approach to local uniformization and its relation to the model theory of valued fields, see [K4].

2. Valuation theoretical preliminaries

2.1. **Some general facts.** For basic facts from valuation theory, see [En], [Ri], [W], [E–P], [Z–S], [K2].

We will denote the algebraic closure of a field K by $\tilde{K}$. Whenever we have a valuation v on K, we will automatically fix an extension of v to the algebraic closure $\tilde{K}$ of K. It does not play a role which extension we choose, except if v is also given on an extension field L of K; in this case, we choose the extension to $\tilde{K}$ to be the restriction of the extension to $\tilde{L}$. We say that v is **trivial** on K if $vK=\{0\}$. If the valuation v of L is trivial on the subfield K, then we may assume that K is a subfield of Lv and the residue map $K\ni a\mapsto av$ is the identity.

We will denote by $K^{\rm sep}$ the separable-algebraic closure of K, and by K^{1/p^∞} its perfect hull. If Γ is an ordered abelian group and p a prime, then we write $\frac{1}{p^\infty}\Gamma$ for the p-divisible hull of Γ,endowed with the unique extension of the ordering from Γ. We leave the easy proof of the following lemma to the reader.

Lemma 2.1. *If K is an arbitrary field and v is a valuation on $K^{\rm sep}$, then $vK^{\rm sep}$ is the divisible hull of vK, and $(Kv)^{\rm sep}\subseteq K^{\rm sep}v$. If in addition v is nontrivial on K, then $K^{\rm sep}v$ is the algebraic closure of Kv.*

Every valuation v on K has a unique extension to K^{1/p^∞}, and it satisfies $vK^{1/p^\infty} = \frac{1}{p^\infty}vK$ and $K^{1/p^\infty}v = (Kv)^{1/p^\infty}$.

For the easy proof of the following lemma, see [B], chapter VI, §10.3, Theorem 1.

Lemma 2.2. *Let $(L|K,v)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i , $i \in I$, are rationally independent over vK, and the residues y_jv, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.*

Moreover, write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J] \tag{8}$$

in such a way that whenever $k \neq \ell$, then there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$. Then

$$vf \;=\; \min_k \, v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k \, vc_k + \sum_{i\in I} \mu_{k,i} vx_i \; . \tag{9}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z} vx_i \\ K(x_i, y_j \mid i \in I, j \in J)v &= Kv\,(y_jv \mid j \in J) \; . \end{aligned}$$

The valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values vx_i and the fact that the residues y_jv, $j \in J$, are algebraically independent over Kv.

The residue map on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the residues y_jv, and the fact that values vx_i , $i \in I$, are rationally independent over vK.

We give two applications of this lemma.

Corollary 2.3. *Take a valued function field $(F|K,v)$ without transcendence defect and set $r = \dim_{\mathbb{Q}} \mathbb{Q} \otimes vF/vK$ and $s = \operatorname{trdeg} Fv|Kv$. Choose elements $x_1, \ldots, x_r$, $y_1, \ldots, y_s \in F$ such that the values $vx_1, \ldots, vx_r$ are rationally independent over vK and the residues $y_1v, \ldots, y_sv$ are algebraically independent over Kv. Then $\mathcal{T} = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ is a transcendence basis of $F|K$. Moreover, vF/vK and the extension $Fv|Kv$ are finitely generated.*

Proof. By the foregoing theorem, the elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ are algebraically independent over K. Since $\operatorname{trdeg} F|K = r + s$ by assumption, these elements form a transcendence basis of $F|K$.

It follows that the extension $F|K(\mathcal{T})$ is finite. By the fundamental inequality (5), this yields that $vF/vK(\mathcal{T})$ and $Fv|K(\mathcal{T})v$ are finite. Since already $vK(\mathcal{T})/vK$ and $K(\mathcal{T})v|Kv$ are finitely generated by the foregoing lemma, it follows that also vF/vK and $Fv|Kv$ are finitely generated. $\square$

Corollary 2.4. *If a valued field extension admits a standard valuation transcendence basis, then it is an extension without transcendence defect.*

Proof. Let $(L|K,v)$ be an extension with standard valuation transcendence basis $\mathcal{T}$, and $F|K$ a subextension of $L|K$ of finite transcendence degree. We have to show that equality holds in (6) for F in place of L. Since $F|K$ is finitely generated, there is a finite subset $\mathcal{T}_0 \subseteq \mathcal{T}$ such that all generators of F are algebraic over $K(\mathcal{T}_0)$. Then $\mathcal{T}_0$ is a standard valuation transcendence basis of $(F(\mathcal{T}_0)|K,v)$, and it follows from Lemma 2.2 that equality holds in (6) for $F' := F(\mathcal{T}_0)$ in place of L. But as $\operatorname{trdeg} F'|K = \operatorname{trdeg} F'|F + \operatorname{trdeg} F|K$, $\operatorname{trdeg} F'v|Kv = \operatorname{trdeg} F'v|Fv + \operatorname{trdeg} Fv|Kv$ and $\dim_{\mathbb{Q}} \mathbb{Q}\otimes vF'/vK = \dim_{\mathbb{Q}} \mathbb{Q}\otimes vF'/vF + \dim_{\mathbb{Q}} \mathbb{Q}\otimes vF/vK$, it follows that

$$\begin{aligned}
\operatorname{trdeg} F'|K &= \operatorname{trdeg} F'|F \,+\, \operatorname{trdeg} F|K \\
&\geq \operatorname{trdeg} F'v|Fv \,+\, \dim_{\mathbb{Q}} \mathbb{Q}\otimes vF'/vF \,+\, \operatorname{trdeg} Fv|Kv \,+\, \dim_{\mathbb{Q}} \mathbb{Q}\otimes vF/vK \\
&= \operatorname{trdeg} F'v|Kv \,+\, \dim_{\mathbb{Q}} \mathbb{Q}\otimes vF'/vK \\
&= \operatorname{trdeg} F'|K\ ,
\end{aligned}$$

hence equality must hold. Since the inequality (6) holds for the two extensions $(F'|F,v)$ and $(F|K,v)$, we find that $\operatorname{trdeg} F|K = \operatorname{trdeg} Fv|Kv + \dim_{\mathbb{Q}} \mathbb{Q}\otimes vF/vK$ must hold. □

Every valued field (L,v) admits a **henselization**, that is, a minimal algebraic extension which is henselian. All henselizations are isomorphic over L, so we will frequently talk of *the* henselization of (L,v), denoted by $(L,v)^h$, or simply L^h. The henselization becomes unique in absolute terms once we fix an extension of the valuation v from L to its algebraic closure. All henselizations are immediate separable-algebraic extensions. They are minimal henselian extensions of (L,v) in the following sense: if (F,v') is a henselian extension field of (L,v), then there is a unique embedding of (L^h,v) in (F,v'). This is the **universal property of the henselization**. We note that every algebraic extension of a henselian field is again henselian.

2.2. **The defect.** Assume that $(L|K,v)$ is a finite extension and the extension of v from K to L is unique (which is always the case when (K,v) is henselian). Then the Lemma of Ostrowski (cf. [En], [Ri], [K2]) says that

$$[L:K] \;=\; (vL:vK)\cdot[Lv:Kv]\cdot p^{\nu} \quad \text{with } \nu\geq 0 \tag{10}$$

where p is the characteristic exponent of Kv. The factor

$$\mathrm{d}(L|K,v) \;:=\; p^{\nu} \;=\; \frac{[L:K]}{(vL:vK)[Lv:Kv]}$$

is called the **defect** of the extension $(L|K,v)$. If $\nu>0$, then we speak of a **nontrivial** defect. If $[L:K]=p$ then $(L|K,v)$ has nontrivial defect if and only if it is immediate. If $\mathrm{d}(L|K,v)=1$, then we call $(L|K,v)$ a **defectless extension**. Note that $(L|K,v)$ is always defectless if $\operatorname{char} Kv=0$. Therefore,

Corollary 2.5. *Every valued field (K,v) with $\operatorname{char} Kv=0$ is a defectless field.*

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. We leave the straightforward proof to the reader.

Lemma 2.6. *Fix an extension of a valuation v from K to its algebraic closure. If $L|K$ and $M|L$ finite extensions and the extension of v from K to M is unique, then*

$$\mathrm{d}(M|K,v) = \mathrm{d}(M|L,v)\cdot \mathrm{d}(L|K,v) \tag{11}$$

In particular, $(M|K,v)$ is defectless if and only if $(M|L,v)$ and $(L|K,v)$ are defectless.

The next lemma follows from Lemma 2.5 of [K8]:

Lemma 2.7. *Take an arbitrary immediate extension $(F|K,v)$ and an algebraic extension $(L|K,v)$ of which every finite subextension admits a unique extension of the valuation and is defectless. Then $F|K$ and $L|K$ are linearly disjoint.*

A valued field (K,v) is called **separably defectless** if equality holds in (5) for every finite separable extension, and it is called **inseparably defectless** if equality holds in (5) for every finite purely in separable extension $L|K$. From the previous lemma, we obtain:

Corollary 2.8. *Every immediate extension of a defectless field is regular. Every immediate extension of an inseparably defectless field is separable.*

The following is an important theorem, as passing to henselizations will frequently facilitate our work.

Theorem 2.9. *Take a valued field (K,v) and fix an extension of v to $\tilde{K}$. Then (K,v) is defectless if and only if its henselization $(K,v)^h$ in $(\tilde{K},v)$ is defectless. The same holds for "separably defectless" and "inseparably defectless" in place of "defectless".*

Proof. For "separably defectless", our assertion follows directly from [En], Theorem (18.2). The proof of that theorem can easily be adapted to prove the assertion for "inseparably defectless" and "defectless". See [K2] for more details. □

Since a henselian field has a unique extension of the valuation to every algebraic extension field, we obtain:

Corollary 2.10. *A valued field (K,v) is defectless if and only if $\mathrm{d}(L|K^h,v)=1$ for every finite extension $L|K^h$.*

Using this corollary together with Lemma 2.6, one easily shows:

Corollary 2.11. *Every finite extension of a defectless field is again a defectless field.*

We will denote by K^r the ramification field of the normal extension $(K^{\mathrm{sep}}|K,v)$, and by K^i its inertia field. As both fields contain the decomposition field of $(K^{\mathrm{sep}}|K,v)$, which is the henselization of K inside of (K^{sep},v), they are henselian.

The following is Proposition 2.8 of [K8]:

Proposition 2.12. *Let (K,v) be a henselian field and N an arbitrary algebraic extension of K within K^r. If $L|K$ is a finite extension, then*

$$\mathrm{d}(L|K,v) \;=\; \mathrm{d}(L.N|N,v)\;.$$

Hence, (K,v) is a defectless field if and only if (N,v) is a defectless field. The same holds for "separably defectless" and "inseparably defectless" in place of "defectless".

Lemma 2.13. *Take a valued field (F,v) and suppose that E is a subfield of F on which v is trivial. Then $E^{\mathrm{sep}} \subset F^i$. Further, if $Fv|Ev$ is algebraic, then $(F.E^{\mathrm{sep}})v = (Fv)^{\mathrm{sep}}$.*

Proof. Our assumption implies that the residue map induces an embedding of E in Fv. By ramification theory ([En], [K2]), $F^i v = (Fv)^{\mathrm{sep}}$. Thus, $(Ev)^{\mathrm{sep}} \subseteq F^i v$. Using Hensel's Lemma, one shows that the inverse of the isomorphism $E \ni a \mapsto av \in Ev$ can be extended from Ev to an embedding of $(Ev)^{\mathrm{sep}}$ in F^i. Its image is separable-algebraically closed and contains E. Hence, $E^{\mathrm{sep}} \subset F^i$. Further, $(F.E^{\mathrm{sep}})v$ contains $E^{\mathrm{sep}}v$, which by Lemma 2.1 contains $(Ev)^{\mathrm{sep}}$. As $F.E^{\mathrm{sep}}|F$ is algebraic, so is $(F.E^{\mathrm{sep}})v|Fv$. Therefore, if $Fv|Ev$ is algebraic, then $(F.E^{\mathrm{sep}})v$ is algebraic over $(Ev)^{\mathrm{sep}}$ and hence separable-algebraically closed. Since $(F.E^{\mathrm{sep}})v \subseteq F^i v = (Fv)^{\mathrm{sep}}$, it follows that $(F.E^{\mathrm{sep}})v = (Fv)^{\mathrm{sep}}$. □

2.3. **Algebraically maximal and separable-algebraically maximal fields.** All algebraically maximal and all separable-algebraically maximal fields are henselian because the henselization is an immediate separable-algebraic extension and therefore these fields must coincide with their henselization. Every henselian defectless field is algebraically maximal. However, the converse is not true in general: algebraically maximal fields need not be defectless (see Example 3.25 in [K9]). But we will see in Corollary 3.12 below that it holds for perfect fields of positive characteristic. More generally, in [K8] it is shown that a valued field of positive characteristic is henselian and defectless if and only if it is algebraically maximal and inseparably defectless. Note that for a valued field of residue characteristic 0, "henselian", "algebraically maximal" and "henselian defectless" are equivalent.

We will assume the reader to be familiar with the theory of pseudo Cauchy sequences as developed in [Ka]. Recall that a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) is **of transcendental type** if it fixes the value of every polynomial $f \in K[X]$, that is, $vf(a_\nu)$ is constant for all large enough $\nu < \lambda$. See [Ka] for the proof of the following theorem.

Theorem 2.14. *A valued field (K,v) is algebraically maximal if and only if every pseudo Cauchy sequence in (K,v) without a limit in K is of transcendental type.*

We also obtain the following result from Kaplansky's main theorems:

Theorem 2.15. *Take an algebraically maximal field (K,v) and two nontrivial immediate extensions $(K(x)|K,v)$ and $(K(y)|K,v)$. If x and y are limits of the same pseudo Cauchy sequence in (K,v) without a limit in K, then $x \mapsto y$ induces a valuation preserving isomorphism from $K(x)$ to $K(y)$.*

Proof. If x and y are limits of the same pseudo Cauchy sequence in (K,v) without a limit in K, then by the foregoing theorem, this pseudo Cauchy sequence is of transcendental type. Our assertion then follows from Theorem 2 of [Ka]. □

We will need the following characterizations of algebraically maximal and separable-algebraically maximal fields; cf. Theorems 1.4, 1.6 and 1.8 of [K8].

Theorem 2.16. *The valued field (K,v) is algebraically maximal if and only if it is henselian and for every polynomial $f \in K[X]$,*

$$\exists x \in K\, \forall y \in K:\ vf(x) \geq vf(y)\ . \tag{12}$$

Similarly, (K,v) is separable-algebraically maximal if and only if (12) holds for every separable polynomial $f\in K[X]$.

3. The algebra of tame and separably tame fields

3.1. **Tame fields.** An algebraic extension of a henselian field is called **purely wild** if it is linearly disjoint from every tame extension. It follows that if $(L|K,v)$ and $(L'|L,v)$ are purely wild extensions, then so is $(L'|K,v)$. We will call (K,v) a **purely wild field** if $(\tilde{K}|K,v)$ is a purely wild extension. For example, for every henselian field (K,v), K^r is a purely wild field (as follows from part b) of Lemma 3.2 or part a) of Lemma 3.4 below). Lemma 2.7 yields important examples of purely wild extensions:

Corollary 3.1. *Every immediate algebraic extension of a henselian field is purely wild.*

The next lemma follows from general ramification theory; see [En], [K2].

Lemma 3.2. *Take a henselian field (K,v).*
a) If $(K_1|K,v)$ and $(K_2|K_1,v)$ are algebraic, then $(K_2|K,v)$ is tame if and only if $(K_1|K,v)$ and $(K_2|K_1,v)$ are.
b) The field K^r is the unique maximal tame extension of (K,v).

Since $K^r|K$ is by definition a separable extension, this lemma yields:

Corollary 3.3. *Every tame extension of a henselian field is separable. Every purely inseparable algebraic extension of a henselian field is purely wild.*

From Lemma 3.2, one easily deduces part a) of the next lemma. Part b) follows from the fact that $L^r=L.K^r$ for every algebraic extension $L|K$.

Lemma 3.4. *a) Let (K,v) be a henselian field. Then (K,v) is a tame field if and only if $K^r=\tilde{K}$. Similarly, (K,v) is a separably tame field if and only if $K^r=K^{\mathrm{sep}}$. Further, (K,v) is a purely wild field if and only if $K^r=K$.*
b) Every algebraic extension of a tame (or separably tame, or purely wild, respectively) field is again a tame (or separably tame, or purely wild, respectively) field.

If (K,v) is a henselian field of residue characteristic 0, then every algebraic extension $(L|K,v)$ is tame, as we have seen in the last section. So we note:

Lemma 3.5. *Every algebraic extension of a henselian field of residue characteristic 0 is a tame extension. Every henselian field of residue characteristic 0 is a tame field.*

From the definition and the fact that every tame extension is separable, we obtain:

Lemma 3.6. *Every tame field is henselian, defectless and perfect.*

In general, infinite algebraic extensions of defectless fields need not again be defectless fields. For example, $\mathbb{F}_p(t)^h$ is a defectless field by Theorem 1.7 and Theorem 2.9, but the perfect hull of $\mathbb{F}_p(t)^h$ is a henselian field admitting an immediate extension generated by a root of the polynomial $X^p-X-\frac{1}{t}$ (cf. Example 3.12 of [K9]). But from Lemmas 3.4 and 3.6 we can deduce that every algebraic extension of a tame field is a defectless field.

The following theorem was proved by M. Pank; cf. [K–P–R].

Theorem 3.7. *Let (K,v) be a henselian field with residue characteristic $p>0$. There exist field complements W_s of K^r in $K^{\rm sep}$ over K, i.e., $K^r.W_s = K^{\rm sep}$ and W_s is linearly disjoint from K^r over K. The perfect hull $W = W_s^{1/p^\infty}$ is a field complement of K^r over K, i.e., $K^r.W = \tilde{K}$ and W is linearly disjoint from K^r over K. The valued fields (W_s,v) can be characterized as the maximal separable purely wild extensions of (K,v), and the valued fields (W,v) are the maximal purely wild extensions of (K,v).*

Moreover, $vW = vW_s$ is the p-divisible hull of vK, and Wv is the perfect hull of Kv; if v is nontrivial, then $Wv = W_sv$.

In [K–P–R], a condition for the uniqueness of these complements is given and its relation to Kaplansky's hypothesis A and the uniqueness of maximal immediate extensions is explained.

We will need the following characterization of purely wild extensions:

Lemma 3.8. *An algebraic extension $(L|K,v)$ of henselian fields of residue characteristic $p>0$ is purely wild if and only if vL/vK is a p-group and $Lv|Kv$ is purely inseparable.*

Proof. By Zorn's Lemma, every purely wild extension is contained in a maximal one. So our assertions on vL/vK and $Lv|Kv$ follows from the corresponding assertions of Theorem 3.7 for vW and Wv.

For the converse, assume that $(L|K,v)$ is an extension of henselian fields of residue characteristic $p>0$ such that vL/vK is a p-group and $Lv|Kv$ is purely inseparable. We have to show that $L|K$ is linearly disjoint from every tame extension $(F|K,v)$. Since every tame extension is a union of finite tame extensions, it suffices to show this under the assumption that $F|K$ is finite. Then $[F:K] = (vF:vK)[Fv:Kv]$. Since p does not divide $(vF:vK)$ and vL/vK is a p-group, it follows that $vF\cap vL = vK$. As $vF+vL \subseteq v(F.L)$, we have that

$$(v(F.L):vL) \;\geq\; ((vF+vL):vL) \;=\; (vF:(vF\cap vL)) \;=\; (vF:vK)\;.$$

Since $Fv|Kv$ is separable and $Lv|Kv$ is purely inseparable, these extensions are linearly disjoint. As $(Fv).(Lv)\subseteq (F.L)v$, we have that

$$[(F.L)v:Lv] \;\geq\; [(Fv).(Lv):Lv] \;=\; [Fv:Kv]\;.$$

Now we compute:

$$[F.L:L] \;\geq\; (v(F.L):vL)[(F.L)v:Lv] \;\geq\; (vF:vK)[Fv:Kv] \;=\; [F:K] \;\geq\; [F.L:L]\;,$$

hence equality holds everywhere. This shows that $L|K$ is linearly disjoint from $F|K$. □

In conjunction with equation (10), this lemma shows:

Corollary 3.9. *The degree of a finite purely wild extension $(L|K,v)$ of henselian fields of residue characteristic $p>0$ is a power of p.*

Using Theorem 3.7, we give some characterizations for tame fields.

Lemma 3.10. *Take a henselian field (K,v) and denote by p the characteristic exponent of Kv. The following assertions are equivalent:*

1) (K,v) is tame,

2) Every purely wild extension $(L|K,v)$ is trivial,

3) (K,v) *is algebraically maximal and closed under purely wild extensions by* p*-th roots,*
4) (K,v) *is algebraically maximal,* vK *is* p*-divisible and* Kv *is perfect.*

Proof. Let (K,v) be a tame field, hence $K^r = \tilde{K}$ by part a) of Lemma 3.4. Then by definition, every purely wild extension of (K,v) must be trivial. This proves 1)⇒2).

Suppose that (K,v) has no purely wild extension. Then in particular, it has no purely wild extension by p-th roots. From Corollary 3.1 we infer that (K,v) admits no proper immediate algebraic extensions, i.e., (K,v) is algebraically maximal. This proves 2)⇒3).

Assume now that (K,v) is an algebraically maximal field closed under purely wild extensions by p-th roots. Take $a \in K$. First, suppose that va is not divisible by p in vK; then the extension $K(b)|K$ generated by an element $b \in \tilde{K}$ with $b^p = a$, together with any extension of the valuation, satisfies $(vK(b):vK) \geq p = [K(b):K] \geq (vK(b):vK)$. Hence, equality holds everywhere, and (5) shows that $(vK(b):vK) = p$ and $K(b)v = Kv$. Hence by Lemma 3.8, $K(b)|K,v)$ is purely wild, contrary to our assumption on (K,v). This shows that vK is p-divisible.

Second, suppose that $va = 0$ and that av has no p-th root in Kv. Then $[K(b)v:Kv] \geq p = [K(b):K] \geq [K(b)v:Kv]$. Hence, equality holds everywhere, and (5) shows that $vK(b) = vK$ and $[K(b)v:Kv] = p$. It follows that $K(b)v|Kv$ is purely inseparable. Again by Lemma 3.8, the extension $(K(b)|K,v)$ is purely wild, contrary to our assumption. This shows that Kv is perfect. Altogether, we have now proved 3)⇒4).

Suppose that (K,v) is an algebraically maximal (and thus henselian) field such that vK is p-divisible and Kv is perfect. Choose a maximal purely wild extension (W,v) in accordance to Theorem 3.7. Together with the last part of Theorem 3.7, our condition on the value group and the residue field yields that $(W|K,v)$ is immediate. But since (K,v) is assumed to be algebraically maximal, this extension must be trivial. This shows that $\tilde{K} = K^r$, i.e., (K,v) is a tame field by part a) of Lemma 3.4. This proves 4)⇒1). □

If the residue field Kv does not admit any finite extension whose degree is divisible by p, then in particular it must be perfect. Hence we can deduce from the previous lemma:

Corollary 3.11. *Every algebraically maximal Kaplansky field is a tame field.*

If K has characteristic $p > 0$, then every extension by p-th roots is purely inseparable and thus purely wild. So the previous lemma together with Lemma 3.6 yields:

Corollary 3.12. *a) A valued field* (K,v) *of characteristic* $p > 0$ *is tame if and only if it is algebraically maximal and perfect.*

b) If (K,v) *is an arbitrary valued field of characteristic* $p > 0$*, then every maximal immediate algebraic extension of its perfect hull is a tame field.*

c) For perfect valued fields of positive characteristic, the properties "algebraically maximal" and "henselian and defectless" are equivalent.

The implication 2)⇒1) of Lemma 3.10 shows:

Corollary 3.13. *Every complement* (W,v) *from Theorem 3.7 is a tame field.*

The next corollary shows how to construct tame fields with suitable prescribed value group and residue field. If $(L|K,v)$ is an extension of valued fields, then a transcendence

basis $\mathcal{T}$ of $L|K$ will be called a **standard valuation transcendence basis** of (L,v) over (K,v) if $\mathcal{T}=\{x_i,y_j\mid i\in I, j\in J\}$ where the values vx_i, $i\in I$, form a maximal set of values in vL rationally independent over vK, and the residues y_jv, $j\in J$, form a transcendence basis of $Lv|Kv$. Note that if $(L|K,v)$ is of finite transcendence degree and admits a standard valuation transcendence basis, then it is an extension without transcendence defect. Note also that the transcendence basis $\mathcal{T}$ given in Theorem 1.8 is a standard valuation transcendence basis.

Corollary 3.14. *Let p be a prime number, Γ a p-divisible ordered abelian group and k a perfect field of characteristic p. Then there exists a tame field K of characteristic p having Γ as its value group and k as its residue field such that $K|\mathbb{F}_p$ admits a standard valuation transcendence basis and the cardinality of K is equal to the maximum of the cardinalities of Γ and k.*

Proof. According to Theorem 2.14. of [K7], there is a valued field (K_0,v) of characteristic p with value group Γ and residue field k, and admitting a standard valuation transcendence basis over its prime field $\mathbb{F}_p$. Now take (K,v) to be a maximal immediate algebraic extension of (K_0,v). Then (K,v) is algebraically maximal, and Lemma 3.10 shows that it is a tame field. Since it is an algebraic extension of (K_0,v), it still admits the same transcendence basis over its prime field. If v is trivial, then $\Gamma=\{0\}$ and $K=k$, whence $|K|=\max\{|\Gamma|,|k|\}$. If v is nontrivial, then K and Γ are infinite and therefore, $|K|=\min\{\aleph_0,|\mathcal{T}|\}\leq\max\{|\Gamma|,|k|\}\leq|K|$, whence again $|K|=\max\{|\Gamma|,|k|\}$. □

Now we will prove an important lemma on tame fields that we will need in several instances.

Lemma 3.15. *Take a tame field (L,v) and a relatively algebraically closed subfield $K\subset L$. If in addition $Lv|Kv$ is an algebraic extension, then K is also a tame field and moreover, vL/vK is torsion free and $Kv=Lv$.*

Proof. The following short and elegant version of the proof was given by Florian Pop. Since (L,v) is tame, it is henselian and perfect. Since K is relatively algebraically closed in L, it is henselian and perfect too. Assume that $(K_1|K,v)$ is a finite purely wild extension; in view of Lemma 3.10, we have to show that it is trivial. By Corollary 3.9, the degree $[K_1:K]$ is a power of p, say p^m. Since K is perfect, $L|K$ and $K_1|K$ are separable extensions. Since K is relatively algebraically closed in L, we know that L and K_1 are linearly disjoint over K. Thus, K_1 is relatively algebraically closed in $K_1.L$, and

$$[K_1.L:L]=[K_1:K]=p^m\;.$$

Since L is assumed to be a tame field, the extension $(K_1.L|L,v)$ must be tame. This implies that

$$(K_1.L)v\mid Lv$$

is a separable extension of degree p^m. By hypothesis, $Lv\mid Kv$ is an algebraic extension, hence also $(K_1.L)v\mid Kv$ and $(K_1.L)v\mid K_1v$ are algebraic. Furthermore, $(K_1.L,v)$ being a henselian field and K_1 being relatively algebraically closed in $K_1.L$, Hensel's Lemma shows that

$$(K_1.L)v\mid K_1v$$

must be purely inseparable. This yields that

$$\begin{aligned} p^m &= [(K_1.L)v : Lv]_{\rm sep} \leq [(K_1.L)v : Kv]_{\rm sep} = [(K_1.L)v : K_1v]_{\rm sep} \cdot [K_1v : Kv]_{\rm sep} \\ &= [K_1v : Kv]_{\rm sep} \leq [K_1v : Kv] \leq [K_1 : K] = p^m \,, \end{aligned}$$

showing that equality holds everywhere, which implies that

$$K_1v \,|\, Kv$$

is separable of degree p^m. Since $K_1|K$ was assumed to be purely wild, we have $p^m = 1$ and the extension $K_1|K$ is trivial.

We have now shown that K is a tame field; hence by Lemma 3.10, vK is p-divisible and Kv is perfect. Since $Lv|Kv$ is assumed to be algebraic, one can use Hensel's Lemma to show that $Lv = Kv$ and that the torsion subgroup of vL/vK is a p-group. But as vK is p-divisible, vL/vK has no p-torsion, showing that vL/vK has no torsion at all. □

A similar fact holds for separably tame fields, as stated in Lemma 3.23 below. Note that the conditions on the residue fields is necessary, even if they are of characteristic 0 (cf. Example 3.9 in [K7]).

The following corollaries will show some nice properties of the class of tame fields. They also admit generalizations to separably tame fields, see Corollary 3.24 below. First we show that a function field over a tame field admits a so-called field of definition which is tame and of finite rank, that is, its value group has only finitely many convex subgroups. This is an important tool in the study of the structure of such function fields.

Corollary 3.16. *For every valued function field F with given transcendence basis $\mathcal{T}$ over a tame field K, there exists a tame subfield K_0 of K of finite rank with $K_0v = Kv$ and vK/vK_0 torsion free, and a function field F_0 with transcendence basis $\mathcal{T}$ over K_0 such that*

$$F = K.F_0 \tag{13}$$

and

$$[F_0 : K_0(\mathcal{T})] = [F : K(\mathcal{T})] \,. \tag{14}$$

Proof. Let $F = K(\mathcal{T})(a_1, \ldots, a_n)$. There exists a finitely generated subfield K_1 of K such that $a_1, \ldots, a_n$ are algebraic over $K_1(\mathcal{T})$ and $[F : K(\mathcal{T})] = [K_1(\mathcal{T})(a_1, \ldots, a_n) : K_1(\mathcal{T})]$. This will still hold if we replace K_1 by any extension field of K_1 within K. As a finitely generated field, (K_1, v) has finite rank. Now let y_j, $j \in J$, be a system of elements in K such that the residues y_jv, $j \in J$, form a transcendence basis of Kv over K_1v. According to Lemma 2.2, the field $K_1(y_j|j \in J)$ has residue field $K_1v(y_jv|j \in J)$ and the same value group as K_1, hence it is again a field of finite rank. Let K_0 be the relative algebraic closure of this field within K. Since by construction, $Kv|K_1v(y_jv|j \in J)$ and thus also $Kv|K_0v$ are algebraic, we can infer from the preceding lemma that K_0 is a tame field with $K_0v = Kv$ and vK/vK_0 torsion free. As an algebraic extension of a field of finite rank, it is itself of finite rank. Finally, the function field $F_0 = K_0(\mathcal{T})(a_1, \ldots, a_n)$ has transcendence basis $\mathcal{T}$ over K_0 and satisfies equations (13) and (14). □

Corollary 3.17. *For every extension $(L|K,v)$ with (L,v) a tame field, there exists a tame intermediate field L_0 such that the extension $(L_0|K,v)$ admits a standard valuation transcendence basis and the extension $(L|L_0,v)$ is immediate.*

Proof. Take $\mathcal{T} = \{x_i, y_j \mid i \in I, j \in J\}$ where the values vx_i, $i \in I$, form a maximal set of values in vL rationally independent over vK, and the residues y_jv, $j \in J$, form a transcendence basis of $Lv|Kv$. With this choice, $vL/vK(\mathcal{T})$ is a torsion group and $Lv|K(\mathcal{T})v$ is algebraic. Let L_0 be the relative algebraic closure of $K(\mathcal{T})$ within L. Then by Lemma 3.15, we have that (L_0,v) is a tame field, that $Lv = L_0v$ and that vL/vL_0 is torsion free and thus, $vL_0 = vL$. This shows that the extension $(L|L_0,v)$ is immediate. On the other hand, $\mathcal{T}$ is a standard valuation transcendence basis of $(L_0|K,v)$ by construction. □

3.2. **Separably tame fields.** Note that "henselian and separably defectless" implies "separable-algebraically maximal".

Since every finite separable-algebraic extension of a separably tame field is tame and thus defectless, a separably tame field is always henselian and separably defectless. The converse is not true; it needs additional assumptions on the value group and the residue field. Under the assumptions that we are going to use frequently, the converse will even hold for "separable-algebraically maximal" in place of "henselian and separably defectless". Before proving this, we need a lemma which makes essential use of Theorem 3.7.

Lemma 3.18. *A henselian field (K,v) is defectless if and only if every finite purely wild extension of (K,v) is defectless. Similarly, (K,v) is separably defectless if and only if every finite separable purely wild extension of (K,v) is defectless.*

Proof. By Theorem 3.7, there exists a field complement W of K^r over K in $K^{\rm sep}$, and W^{1/p^∞} is a field complement of K^r over K in $\tilde{K}$. Consequently, given any finite extension (or finite separable extension, respectively) $(L|K,v)$, there is a finite subextension $N|K$ of $K^r|K$ and a finite subextension (or finite separable subextension, respectively) $W_0|K$ of $W^{1/p^\infty}|K$ (or of $W|K$, respectively) such that $L \subseteq N.W_0$. If $(N.W_0|K,v)$ is defectless, then so is $(L|K,v)$ by Lemma 2.6. Hence (K,v) is defectless (or separably defectless, respectivel) if and only if every such extension $(N.W_0|K,v)$ is defectless. Since $(N|K,v)$ is a tame extension, Lemma 2.12 shows that

$$d(N.W_0|N,v) = d(W_0|K,v)\;.$$

Hence, $(L|K,v)$ is defectless if $(W_0|K,v)$ is defectless. This yields our assertion. □

An **Artin-Schreier extension** of a field K of characteristic $p > 0$ is an extension generated by a root of a polynomial of the form $X^p - X - a$ with $a \in K$.

Lemma 3.19. *Take a valued field (K,v) of characteristic $p > 0$. The following assertions are equivalent:*
1) (K,v) is separably tame,
2) Every separable purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is separable-algebraically maximal and closed under purely wild Artin-Schreier extensions,
4) (K,v) is separable-algebraically maximal, vK is p-divisible and Kv is perfect.

Proof. Let (K,v) be a separably tame field. Then by definition, every separable purely wild extension of (K,v) must also be tame, hence trivial. This proves 1)⇒2).

Now suppose that 2) holds. Then in particular, (K,v) admits no purely wild Artin-Schreier extensions. Furthermore, (K,v) admits no proper separable-algebraic immediate extensions, as they would be purely wild. Consequently, (K,v) is separable-algebraically maximal. This proves 2)⇒3).

If (K,v) is closed under purely wild Artin-Schreier extensions, then vK is p-divisible and Kv is perfect (cf. Corollary 2.17 of [K7]). This proves 3)⇒4).

Suppose that (K,v) is a separable-algebraically maximal field such that vK is p-divisible and Kv is perfect. Then (K,v) is henselian. Choose a maximal separable purely wild extension (W_s,v) in accordance to Theorem 3.7. Our condition on the value group and the residue field yields that $(W_s|K,v)$ is immediate. But since (K,v) is assumed to be separable-algebraically maximal, this extension must be trivial. This shows that $K^{\mathrm{sep}} = K^r$, i.e., (K,v) is a separably tame field by part a) of Lemma 3.4. This proves 4)⇒1). □

Suppose that (K,v) separably tame. Choose (W_s,v) according to Theorem 3.7. Then by condition 2) of the lemma above, the extension $(W_s|K,v)$ must be trivial. This yields that $(K^{1/p^\infty},v)$ is the unique maximal purely wild extension of (K,v). Further, (K,v) also satisfies condition 3) of the lemma. From Corollary 4.6 of [K8] it follows that (K,v) is dense in $(K^{1/p^\infty},v)$, i.e., K^{1/p^∞} lies in the completion of (K,v). This proves:

Corollary 3.20. *If (K,v) is separably tame, then the perfect hull K^{1/p^∞} of K is the unique maximal purely wild extension of (K,v) and lies in the completion of (K,v). In particular, every immediate algebraic extension of a separably tame field (K,v) is purely inseparable and included in the completion of (K,v).*

Lemma 3.21. *(K,v) is a separably tame field if and only if $(K^{1/p^\infty},v)$ is a tame field. Consequently, if $(K^{1/p^\infty},v)$ is a tame field, then (K,v) is dense in $(K^{1/p^\infty},v)$.*

Proof. Suppose that (K,v) is a separably tame field. Then by the maximality stated in the previous corollary, $(K^{1/p^\infty},v)$ admits no proper purely wild algebraic extensions. Hence by Lemma 3.10, $(K^{1/p^\infty},v)$ is a tame field.

For the converse, suppose that $(K^{1/p^\infty},v)$ is a tame field. Observe that the extension $(K^{1/p^\infty}|K,v)$ is purely wild and contained in every maximal purely wild extension of (K,v). Consequently, if $(K^{1/p^\infty},v)$ admits no purely wild extension at all, then $(K^{1/p^\infty},v)$ is the unique maximal purely wild extension of (K,v). Then in view of Theorem 3.7, K^{1/p^∞} must be a field complement for K^r over K in $\tilde{K}$. This yields that $K^r = K^{\mathrm{sep}}$, hence by part b) of Lemma 3.2, $(K^{\mathrm{sep}}|K,v)$ is a tame extension, showing that (K,v) is a separably tame field. By Corollary 3.20, it follows that (K,v) is dense in $(K^{1/p^\infty},v)$. □

The following lemma describes the interesting behaviour of separably tame fields under composition of places.

Lemma 3.22. *Take a separably tame field (K,v) of characteristic $p>0$ and let P be the place associated with v. Assume that $P = P_1P_2P_3$ where P_1 is a coarsening of P, P_2 is a place on KP_1 and P_3 is a place on KP_1P_2. Assume further that P_2 is nontrivial (but P_1 and P_3 may be trivial). Then (KP_1,P_2) is a separably tame field. If also P_1 is nontrivial, then (KP_1,P_2) is a tame field.*

Proof. By Lemma 3.10, vK is p-divisible. The same is then true for $v_{P_2}(KP_1)$. We wish to show that the residue field KP_1P_2 is perfect. Indeed, assume that this were not the case. Then there is an Artin-Schreier extension of (K, P_1P_2) which adjoins a p-th root to the residue field KP_1P_2 (cf. Lemma 2.13 of [K7]). Since this residue field extension is purely inseparable, the induced extension of the residue field $Kv = KP_1P_2P_3$ can not be separable of degree p. This shows that the Artin-Schreier extension is a separable purely wild extension of (K, v), contrary to our assumption on (K, v).

By Lemma 3.19, (K, P) is separable-algebraically maximal. This yields that the same is true for (K, P_1P_2); indeed, if $(L|K, P_1P_2)$ is an immediate extension, then $LP_1P_2 = KP_1P_2$, whence $LP_1P_2P_3 = KP_1P_2P_3$, showing that also $(L|K, P)$ is immediate. If P_1 is trivial (hence w.l.o.g. equal to the identity map), then $(KP_1, P_2) = (K, P_1P_2)$ is separable-algebraically maximal, and it follows from Lemma 3.19 that (KP_1, P_2) is a separably tame field.

Now assume that P_1 is nontrivial. Suppose that there is a nontrivial immediate algebraic extension of (KP_1, P_2). Choose an element $b \notin KP_1$ in this extension, and let g be its minimal polynomial. Choose a monic polynomial $f \in K[X]$ such that $fP_1 = g$, and a root a of f. Then there is an extension of P_1 to $K(a)$ such that $aP_1 = b$. It follows from the fundamental inequality that $K(a)P_1 = KP_1(b)$ and that $(K(a), P_1)$ and (K, P_1) have the same value group. But as $(KP_1(b)|KP_1, P_2)$ is immediate, it now follows that also $(K(a)|K, P_1P_2P_3)$ is immediate. Note that we can always choose f to be separable as we may add a summand cX with $v_{P_1}c > 0$, which does not change the image of f under P_1. In this way, we obtain a contradiction to the fact that (K, P) is separable-algebraically maximal. We have thus shown that (KP_1, P_2) is an algebraically maximal field, and it follows from Lemma 3.10 that (KP_1, P_2) is a tame field. □

The following is an analogue of Lemma 3.15.

Lemma 3.23. *Let (K, v) be a separably tame field and $k \subset K$ a relatively algebraically closed subfield of K. If the residue field extension $Kv|kv$ is algebraic, then (k, v) is also a separably tame field.*

Proof. Since k is relatively algebraically closed in K, it follows that also k^{1/p^∞} is relatively algebraically closed in K^{1/p^∞}. Since (K, v) is a separably tame field, $(K^{1/p^\infty}, v)$ is a tame field by Lemma 3.21. From this lemma we also know that $Kv = K^{1/p^\infty}v$ and $vK = vK^{1/p^\infty}$. Our assumption on $Kv\,|\,kv$ yields that the extension $K^{1/p^\infty}v\,|\,k^{1/p^\infty}v$ is algebraic. From Lemma 3.15 we can now infer that $(k^{1/p^\infty}, v)$ is a tame field with $k^{1/p^\infty}v = K^{1/p^\infty}v = Kv$ and $vK^{1/p^\infty}/vk^{1/p^\infty} = vK/vk^{1/p^\infty}$ torsion free. Again by Lemma 3.21, (k, v) is thus a separably tame field with $kv = k^{1/p^\infty}v = Kv$ and $vK/vk = vK/vk^{1/p^\infty}$ torsion free. □

Corollary 3.24. *Corollary 3.16 also holds for separably tame fields in place of tame fields. More precisely, if $F|K$ is a separable extension, then F_0 and K_0 can be chosen such that $F_0|K_0$ is a separable extension. Moreover, if vK is cofinal in vF then it can also be assumed that vK_0 is cofinal in vF_0.*

Proof. Since the proof of Corollary 3.16 only involves Lemma 3.15, it can be adapted by use of Lemma 3.23. The first additional assertion can be shown using the fact that the

finitely generated separable extension $F|K$ is separably generated. The second additional assertion is seen as follows. If vF admits a biggest proper convex subgroup, then let K_0 contain a nonzero element whose value does not lie in this subgroup. If vF and thus also vK does not admit a biggest proper convex subgroup, then first choose F_0 and K_0 as in the (generalized) proof of Lemma 3.16; since F_0 has finite rank, there exists some element in K whose value does not lie in the convex hull of vF_0 in vF, and adding this element to K_0 and F_0 will make vK_0 cofinal in vF_0. □

With the same proof as for Corollary 3.17, but using Lemma 3.23 in place of Lemma 3.15, one shows:

Corollary 3.25. *Corollary 3.17 also holds for separably tame fields in place of tame fields.*

4. Model theoretic preliminaries

We will now discuss the axiomatization of valued fields and some of their important properties. A valuation v on a field K can be given in several ways. We can take the **valuation divisibility relation** and formalize it as a binary predicate R_v which in every valued field is to be interpreted as

$$R_v(x,y) \iff vx \geq vy \;.$$

But we can also take the valuation ring and formalize it as a predicate $\mathcal{O}$ which in every valued field (K,v) is to be interpreted as

$$\mathcal{O}(x) \iff x \in \mathcal{O} \;.$$

This predicate can be defined from the valuation divisibility relation by

$$\mathcal{O}(x) \;\leftrightarrow\; R_v(x,1) \;.$$

If we are working in the language of fields (what we usually do), then the valuation divisibility relation can be defined from the predicate $\mathcal{O}$ by

$$R_v(x,y) \;\leftrightarrow\; (y \neq 0 \,\wedge\, \mathcal{O}(xy^{-1})) \,\vee\, x=0 \;,$$

whereas in general, it can not be defined using $\mathcal{O}$ and the language of rings without the use of quantifiers, as in

$$R_v(x,y) \;\leftrightarrow\; (\exists z\; yz=1 \,\wedge\, \mathcal{O}(xz)) \,\vee\, x=0 \;.$$

This fact is only of importance for questions of quantifier elimination, and only if one has decided to work in the language of rings. Note that two fields are equivalent in the language of rings if and only if they are equivalent in the language of fields. A similar assertion holds for valued fields in the respective languages, and it also holds for the notions "elementary extension" and "existentially closed in" in place of "equivalent".

We prefer to write "$vx \geq vy$" in place of "$R_v(x,y)$". For convenience, we define the following relations:

$$\begin{aligned} vx > vy \quad &\leftrightarrow \quad vx \geq vy \,\wedge\, \neg(vy \geq vx) \\ vx = vy \quad &\leftrightarrow \quad vx \geq vy \,\wedge\, vy \geq vx \;. \end{aligned}$$

The definitions for the reversed relations $vx \leq vy$ and $vx < vy$ are obvious.

We will work in the language $\mathcal{L}_{\rm VF}$ of valued fields as introduced in the introduction. The **theory of valued fields** is the theory of fields (in the language $\mathcal{L}_{\rm F}$) together with the axioms

(V0) $\quad (\forall y\, vx \geq vy) \;\Leftrightarrow\; x = 0$

(VT) $\quad v(x-y) \geq vx \;\vee\; v(x-y) \geq vy$

and the axioms which state that the value group is an ordered abelian group:

(VV$\not R$) $\quad \neg(vx < vx)$

(VVT) $\quad vx < vy \wedge vy < vz \;\Rightarrow\; vx < vz$

(VVC) $\quad vx < vy \vee vx = vy \vee vx > vy$

(VVG) $\quad vx < vy \;\Rightarrow\; vxz < vyz$

(the group axioms for the value group follow from the group axioms for the multiplicative group of the field).

The following facts are well-known; the easy proofs are left to the reader.

Lemma 4.1. *Take a valued field (K,v).*

a) For every sentence φ in the language of ordered groups there is a sentence φ' in the language of valued fields such that for every valued field (K,v), φ holds in vK if and only if φ' holds in (K,v).

b) For every sentence φ in the language of rings there is a sentence φ' in the language of valued fields such that for every valued field (K,v), φ holds in Kv if and only if φ' holds in (K,v).

As immediate consequences of this lemma, we obtain:

Corollary 4.2. *If (K,v) and (L,v) are valued fields such that $(K,v) \equiv (L,v)$ in the language of valued fields, then $vK \equiv vL$ in the language of ordered groups, and $Kv \equiv Lv$ in the language of rings (and thus also in the language of fields). The same holds with $\prec$ or $\prec_\exists$ in place of $\equiv$.*

Corollary 4.3. *If (K,v) is κ-saturated, then so are vK (in the language of ordered groups) and Kv (in the language of fields).*

The property of being henselian is axiomatized by the following axiom scheme:

(HENS)
$$\begin{aligned} & vy \geq 0 \wedge \textstyle\bigwedge_{1\leq i\leq n} vx_i \geq 0 \wedge v(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n) > 0 \\ & \qquad\qquad\qquad\qquad \wedge\; v(ny^{n-1} + (n-1)x_1y^{n-2} + \ldots + x_{n-1}) = 0 \\ & \Rightarrow\; \exists z\; v(y-z) > 0 \wedge z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n = 0 \qquad (n \in \mathbb{N})\,. \end{aligned}$$

Here we use one of the forms of Hensel's Lemma to characterize henselian fields (see [K2] for an extensive collection). In view of Theorem 2.16, also the property of being algebraically maximal is easily axiomatized by axiom scheme (HENS) together with the following axiom scheme:

(MAXP)
$$\exists y \forall z : v(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n) \geq v(z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n) \qquad (n \in \mathbb{N})\,.$$

By the same theorem, the property of being separable-algebraically maximal is axiomatized by axiom scheme (HENS) together with a version of axiom scheme (MAXP) restricted to separable polynomials. This is obtained by adding sentences that state that

the coefficient of at least one power y^i for $i > 0$ not divisible by the characteristic of the field is nonzero.

The following was proved by Delon [D] and Ershov [Er]. For the case of valued fields of positive characteristic, we give an alternative proof in [K8].

Lemma 4.4. *The property of being henselian and defectless is elementary.*

5. The AKE$^\exists$ Principle

5.1. **Necessary conditions for the AKE$^\exists$ Principle.** In this section we discuss tools for the proof of AKE$^\exists$ Principles and ask for those properties that a valued field must have if it is an AKE$^\exists$-field.

We will need a model theoretic tool which we will apply to valued fields as well as value groups and residue fields. We consider a countable language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathfrak{B}$ and $\mathfrak{A}^*$ with a common substructure $\mathfrak{A}$. We will say that σ is an **embedding of $\mathfrak{B}$ in $\mathfrak{A}^*$ over $\mathfrak{A}$** if it is an embedding of $\mathfrak{B}$ in $\mathfrak{A}^*$ that leaves the universe A of $\mathfrak{A}$ elementwise fixed.

Proposition 5.1. *Let $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \subset \mathfrak{A}^*$ be extensions of $\mathcal{L}$-structures. If $\mathfrak{B}$ embeds over $\mathfrak{A}$ in $\mathfrak{A}^*$ and if $\mathfrak{A} \prec_\exists \mathfrak{A}^*$, then $\mathfrak{A} \prec_\exists \mathfrak{B}$. Conversely, if $\mathfrak{A} \prec_\exists \mathfrak{B}$ holds and if $\mathfrak{A}^*$ is $|B|^+$-saturated, then $\mathfrak{B}$ embeds over $\mathfrak{A}$ in $\mathfrak{A}^*$.*

Proof. Since $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and of $\mathfrak{A}^*$, both $(\mathfrak{B}, A)$ and $(\mathfrak{A}^*, A)$ are $\mathcal{L}(A)$-structures.

Suppose that σ is an embedding of $\mathfrak{B}$ over $\mathfrak{A}$ in $\mathfrak{A}^*$. Then every $\mathcal{L}(A)$-sentence will hold in $(\mathfrak{B}, A)$ if and only if it holds in $(\sigma\mathfrak{B}, A)$ (because isomorphic structures are equivalent). Every existential $\mathcal{L}(A)$-sentence φ which holds in $(\mathfrak{B}, A)$ will then also hold in $(\mathfrak{A}^*, A)$ since $\mathfrak{A}^*$ is an extension of $\sigma\mathfrak{B}$. If in addition $\mathfrak{A} \prec_\exists \mathfrak{A}^*$, then φ will also hold in $(\mathfrak{A}, A)$. This proves our first assertion.

Now suppose that $\mathfrak{A} \prec_\exists \mathfrak{B}$. Then every $\mathcal{L}(A)$-sentence which holds in $(\mathfrak{B}, A)$ also holds in $(\mathfrak{A}, A)$ and, as $(\mathfrak{A}^*, A)$ is an extension of $(\mathfrak{A}, A)$, also in $(\mathfrak{A}^*, A)$. Now assume in addition that $\mathfrak{A}^*$ is $|B|^+$-saturated. Since $|A| \leq |B| < |B|^+$, also $(\mathfrak{A}^*, A)$ is $|B|^+$-saturated. Hence by Lemma 5.2.1. of [C–K], $(\mathfrak{B}, A)$ embeds in $(\mathfrak{A}^*, A)$, i.e., $\mathfrak{B}$ embeds in $\mathfrak{A}^*$ over $\mathfrak{A}$. □

If we have an extension $\mathfrak{A} \subseteq \mathfrak{B}$ of $\mathcal{L}$-structures and want to show that $\mathfrak{A} \prec_\exists \mathfrak{B}$, then by our lemma it suffices to show that $\mathfrak{B}$ embeds over $\mathfrak{A}$ in some elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$. This is the motivation for **embedding lemmas**, which will play an important role later in our paper. When we look for such embeddings, we can use a very helpful principle which follows immediately from the previous proposition because $\mathfrak{A} \prec_\exists \mathfrak{B}$ holds if and only if $\mathfrak{A} \prec_\exists \mathfrak{B}_0$ for every substructure $\mathfrak{B}_0$ of $\mathfrak{B}$ which is finitely generated over $\mathfrak{A}$ (as every existential sentence only talks about finitely many elements).

Lemma 5.2. *Let $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{A}^*$ be extensions of $\mathcal{L}$-structures. Assume that $\mathfrak{A}^*$ is $|B|^+$-saturated. If every substructure of $\mathfrak{B}$ which is finitely generated over $\mathfrak{A}$ embeds over $\mathfrak{A}$ in $\mathfrak{A}^*$, then also $\mathfrak{B}$ embeds over $\mathfrak{A}$ in $\mathfrak{A}^*$.*

We will also need the following well known facts (which were proved, e.g., in L. van den Dries' thesis).

Lemma 5.3. *a) Take an extension $G|H$ of torsion free abelian groups. consider it as an extension of $\mathcal{L}_{\mathrm{G}}$-structures. If H is existentially closed in G in the language $\mathcal{L}_{\mathrm{G}} = \{+,-,0\}$ of groups, then G/H is torsion free.*
b) Take a field extension $L|K$. If K is existentially closed in L in the language $\mathcal{L}_{\mathrm{F}}$ of fields (or in the language $\mathcal{L}_{\mathrm{R}}$ of rings), then $L|K$ is regular.

An immediate consequence of the AKE$^\exists$ Principle (3) is the following observation:

Lemma 5.4. *Every AKE$^\exists$-field is algebraically maximal.*

Proof. Take a valued field (K,v) which admits an immediate algebraic extension (L,v). Then by Lemma 5.3 b), K is not existentially closed in L. Hence, (K,v) is not existentially closed in (L,v). But $vK = vL$ and $Kv = Lv$, so that the conditions $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$ hold. This shows that (K,v) is not an AKE$^\exists$-field. □

In particular, this lemma shows that every AKE$^\exists$-field must be henselian.

A special case of the AKE$^\exists$ Principle is given if an extension $(L|K,v)$ is immediate. Then, the side conditions are trivially satisfied. We conclude that an AKE$^\exists$-field must in particular be existentially closed in every immediate extension (L,v). (We have used this idea already in the proof of the foregoing lemma.) We can exploit this fact by taking (M,v) to be a maximal immediate extension of (K,v), to see which properties of (M,v) are inherited by (K,v) if $(K,v) \prec_\exists (M,v)$. We know that (M,v) has strong structural properties: every pseudo Cauchy sequence has a limit (cf. [Ka]), and it is spherically complete (cf. [Ku2]).

Maximal Kaplansky fields are isomorphic to power series fields (possibly with nontrivial factor sets, cf. [Ka]). Since (M,v) must coincide with its henselization, which is an immediate extension, it is henselian. By Theorem of [W] (M,v) is also a defectless field. Nevertheless, if (K,v) is henselian of residue characteristic 0, then $(K,v) \prec (\mathfrak{M},v)$, which means that the elementary properties of (M,v) are not stronger than those of (K,v). For other classes of valued fields, the situation can be very different. Let us prove that every AKE$^\exists$-field is henselian and defectless:

Lemma 5.5. *Let (K,v) be a valued field and assume that there is some maximal immediate extension (M,v) of (K,v) which satisfies $(K,v) \prec_\exists (M,v)$. Then (K,v) is henselian and defectless. In particular, every AKE$^\exists$-field is henselian and defectless.*

Proof. Let $(E|K,v)$ be an arbitrary finite extension. Working in the language of valued field augmented by an additional predicate for a subfield, we take $(E|K,v)^*$ to be a $|M|^+$-saturated elementary extension of $(E|K,v)$. Then (E^*,v^*) and (K^*,v^*) are $|M|^+$-saturated elementary extensions of (E,v) and (K,v) respectively. Since by assumption (K,v) is existentially closed in (M,v), Proposition 5.1 shows that we can embed (M,v) over (K,v) in (K^*,v^*). We identify it with its image in (K^*,v^*). Since $(E^*|K^*,v^*)$ is an elementary extension of $(E|K,v)$, the extensions $E|K$ and $K^*|K$ are linearly disjoint. Therefore, $n := [E:K] = [E.M:M]$.

We will prove that the extension $(E.M,v^*)|(E,v)$ is immediate. Since $E.M|M$ is algebraic and $vM = vK$, we know from the fundamental inequality (5) that $v^*(E.M)/vK$ and hence also $v^*(E.M)/vE$ is a torsion group. For the same reason, $Mv = Kv$ yields that $(E.M)v^*|Kv$ and hence also $(E.M)v^*|Ev$ is algebraic. On the other hand, since (E^*,v^*)

is an elementary extension of (E,v) we know by Lemma 5.3 that v^*E^*/vE is torsion free and that Ev is relatively algebraically closed in E^*v. Combining these facts, we get that

$$v^*(E.M) \;=\; vE \quad\text{and}\quad (E.M)v^* \;=\; Ev\;,$$

showing that $(E.M,v^*)|(E,v)$ is immediate, as contended.

Since (M,v) is maximal, it is a henselian and defectless field, as we have mentioned above. Consequently,

$$[E:K]=n=[E.M:M]=(v^*(E.M):vM)\cdot[(E.M)v^*:Mv]=(vE:vK)\cdot[Ev:Kv]\;,$$

which shows that $(E|K,v)$ is defectless and that the extension of the valuation v from K to E is unique. Since (E,v) was an arbitrary finite extension of (K,v), this shows that (K,v) is a henselian and defectless field. □

5.2. **Extensions without transcendence defect.** Our first goal in this section is to prove Theorem 1.2. Take a henselian and defectless field (K,v) and an extension $(L|K,v)$ without transcendence defect. We choose (K^*,v^*) to be an $|L|^+$-saturated elementary extension of (K,v). Since "henselian" is an elementary property, (K^*,v^*) is henselian like (K,v). Further, it follows from Corollary 4.3 that K^*v^* is an $|Lv|^+$-saturated elementary extension of Kv and that v^*K^* is a $|vL|^+$-saturated elementary extension of vK. Assume that the side conditions $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$ hold. Then by Proposition 5.1, there exist embeddings

$$\rho:\; vL \longrightarrow v^*K^*$$

over vK and

$$\sigma:\; Lv \longrightarrow K^*v^*$$

over Kv. Here, the embeddings of value groups and residue fields are understood to be monomorphisms of groups and fields, respectively.

We wish to prove that $(K,v) \prec_\exists (L,v)$. By Proposition 5.1, this can be achieved by showing the existence of an embedding

$$\iota:\; (L,v) \longrightarrow (K^*,v^*)$$

over K, i.e., an embedding of L in K^* over K preserving the valuation, that is,

$$\forall x\in L:\; x\in\mathcal{O}_L \Longleftrightarrow \iota x\in\mathcal{O}_{K^*}\;.$$

According to Lemma 5.2, such an embedding exists already if it exists for every finitely generated subextension $(F|K,v)$ of $(L|K,v)$. In this way, we reduce our embedding problem to an embedding problem for valued algebraic function fields $(F|K,v)$. Since in the present case, $(L|K,v)$ is assumed to be an extension without transcendence defect, the same holds for every finitely generated subextension $(F|K,v)$. The case of such valued function fields is covered by the following embedding lemma.

Lemma 5.6. (Embedding Lemma I)
Let (K,v) be a defectless field (the valuation is allowed to be trivial), $(F|K,v)$ a valued function field without transcendence defect and (K^*,v^*) a henselian extension of (K,v). Assume that vF/vK is torsion free and that $Fv|Kv$ is separable. If $\rho:\; vF \longrightarrow v^*K^*$ is an embedding over vK and $\sigma:\; Fv \longrightarrow K^*v^*$ is an embedding over Kv, then there exists an embedding $\iota:(F,v)\longrightarrow(K^*,v^*)$ over (K,v) that respects ρ and σ, i.e., $v^*(\iota a)=\rho(va)$ and $(\iota a)v^*=\sigma(av)$ for all $a\in F$.

Proof. We choose a transcendence basis $\mathcal{T}$ as in Theorem 1.8. First we will construct the embedding for $K(\mathcal{T})$ and then we will show how to extend it to F.

We choose elements $x'_1,\ldots,x'_r \in K^*$ such that $v^*x'_i = \rho(vx_i)$, $1\leq i\leq r$. The values $v^*x'_1,\ldots,v^*x'_r$ are rationally independent over vK since the same holds for their preimages $vx_1,\ldots,vx_r$ and this property is preserved by every monomorphism over vK. Next, we choose elements $y'_1,\ldots,y'_s \in \mathcal{O}_{K^*}^\times$ such that $y'_jv^* = \sigma(y_jv)$, $1\leq j\leq s$. The residues $y'_1v^*,\ldots,y'_sv^*$ are algebraically independent over Kv since the same holds for their preimages $y_1v,\ldots,y_sv$ and this property is preserved by every monomorphism over Kv. Consequently, the elements $x'_1,\ldots,x'_r$ and $y'_1,\ldots,y'_s$ as well as the elements $x_1,\ldots,x_r$ and $y_1,\ldots,y_s$ satisfy the conditions of Lemma 2.2. Hence, both sets $\mathcal{T}$ and $\mathcal{T}'$ are algebraically independent over K, so that the assignment

$$x_i\mapsto x'_i\;,\; y_j\mapsto y'_j\;\; 1\leq i\leq r\,,\; 1\leq j\leq s$$

induces an isomorphism $\iota : K(\mathcal{T}) \longrightarrow K(\mathcal{T}')$. Furthermore, for every $f\in K[\mathcal{T}]$, written as in (8),

$$\begin{aligned} v^*(\iota f) &= \min_k \left(v^*c_k + \sum_{1\leq i\leq r} \mu_{k,i} v^*x'_i \right) = \min_k \left(vc_k + \sum_{1\leq i\leq r} \mu_{k,i}\rho vx_i \right) \\ &= \rho \min_k \left(vc_k + \sum_{1\leq i\leq r} \mu_{k,i} vx_i \right) = \rho(vf)\;, \end{aligned}$$

showing that ι respects the restriction of ρ to $vK(\mathcal{T})$. If $vf=0$, then

$$fv = \left(\sum_\ell c_\ell \prod_{1\leq j\leq s} y_j^{\nu_{\ell,j}} \right) v = \sum_\ell (c_\ell v) \prod_{1\leq j\leq s} (y_jv)^{\nu_{\ell,j}}$$

where the sum runs only over those $\ell=k$ for which $\mu_{k,i}=0$ for all i, and a similar formula holds for $(\iota f)v$ with the same indices ℓ. Hence,

$$\begin{aligned} (\iota f)v^* &= \sum_\ell (c_\ell v^*) \prod_{1\leq j\leq s} (y_jv^*)^{\nu_{\ell,j}} = \sum_\ell (c_\ell v) \prod_{1\leq j\leq s} \sigma(y_jv)^{\nu_{\ell,j}} \\ &= \sigma \left(\sum_\ell (c_\ell v) \prod_{1\leq j\leq s} (y_jv)^{\nu_{\ell,j}} \right) = \sigma(fv)\;, \end{aligned}$$

showing that ι respects the restriction of σ to $K(\mathcal{T})v$.

To simplify notation, we will write $F_0 = K(\mathcal{T})$. We will now construct a valuation preserving embedding of the henselization F_0^h over K in (K^*,v^*). The restriction of this embedding is the required embedding of F. Observe that F_0^h contains the henselization $K(\mathcal{T})^h$. By the universal property of henselizations, ι extends to a valuation preserving embedding of F_0^h in K^* since by hypothesis, K^* is henselian. Since $F_0^h|F_0$ is immediate, this embedding also respects the above mentioned restrictions of ρ and σ. Through this embedding, we will from now on identify F_0^h with its image in K^*.

Now we have to extend ι (which by our identification has become the identity) to an embedding of F^h in K^* (over F_0^h) which respects ρ and σ. This is done as follows. By

hypothesis and our choice of $\mathcal{T}$, the extension $F^h|F_0^h$ is finite and tame with $vF^h = vF = vF_0 = vF_0^h$. Consequently, $F^hv|F_0^hv$ is a finite separable extension, hence generated by one element, say av with $a \in F_0^h$. Take a monic polynomial $f \in \mathcal{O}_{F_0^h}[X]$ whose residue polynomial fv is the minimal polynomial of av over F_0^hv; by hypothesis, fv is separable. Hensel's Lemma shows that there exists exactly one root a of f in F^h having residue av, and exactly one root a' of f in the henselian field K^* having residue $\sigma(av)$. The assignment

$$a \mapsto a'$$

induces an isomorphism $\iota : F_0^h(a) \longrightarrow F_0^h(a')$ which is valuation preserving since F_0^h is henselian. As $vF^h = vF_0^h$, we also have that $vF_0^h(a) = vF_0^h$. Thus, ι respects ρ (which after the above identification is the identity). We have to show that ι also respects σ.

Let $n = [F_0^h(a) : F_0^h]$. Since the elements $1, av, \ldots, (av)^{n-1}$ are linearly independent, the basis $1, a, \ldots, a^{n-1}$ is a valuation basis of $F_0^h(a)|F_0^h$, that is,

$$v \sum_{i=0}^{n-1} c_i a^i \;=\; \min_i vc_i$$

for any choice of $c_i \in F_0^h$. Take $g(a) \in F_0^h[a]$ where $g \in F_0^h[X]$ is of degree $< n$; if $vg(a) = 0$, then $g \in \mathcal{O}_{F_0^h}[X]$ and thus, $g(a)v = (gv)(av)$. In this case,

$$(\iota g(a))v^* = g(a')v^* = (gv)(a'v^*) = (gv)(\sigma(av)) = \sigma((gv)(av)) = \sigma(g(a)v)\;.$$

This proves that ι respects σ.

We have constructed an embedding of $F_0^h(a)$ in K^* which respects ρ and σ. But since $F^h|F_0^h$ is a finite tame extension with $vF^h = vF_0^h$, we have:

$$[F^h : F_0^h] = [F^hv : F_0^hv] = [F_0^h(a)v : F_0^hv] = [F_0^h(a) : F_0^h]$$

which shows that $F^h = F_0^h(a)$. Hence, ι is the required embedding. □

We return to the proof of Theorem 1.2. We take any finitely generated subextension $F|K$ of $L|K$. As pointed out above, $(F|K, v)$ is an extension without transcendence defect. By assumption, $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$, which implies that $vK \prec_\exists vF$ and $Kv \prec_\exists Fv$ because $vF|vK$ is a subextension of $vL|vK$, and $Fv|Kv$ is a subextension of $Lv|Kv$. So we can infer from Lemma 5.3 that the conditions "vF/vK is torsion free" and "$Fv|Kv$ is separable" are satisfied. Hence there is an embedding

$$\iota :\; (F, v) \longrightarrow (K^*, v^*)$$

over K that respects the restriction of ρ to vF and the restriction of σ to Fv. Since this holds for every finitely generated subextension $(F|K, v)$ of $(L|K, v)$, it follows from Lemma 5.2 that also (L, v) embeds in (K^*, v^*) over K. By Proposition 5.1, this shows that (K, v) is existentially closed in (L, v), and we have now proved Theorem 1.2.

For further use, we have to make our result more precise:

Lemma 5.7. (Embedding Lemma II)
Take a defectless field (K, v) (the valuation is allowed to be trivial), an extension $(L|K, v)$ without transcendence defect and an $|L|^+$-saturated henselian extension (K^*, v^*) of (K, v).

Assume that vL/vK is torsion free and that $Lv|Kv$ is separable. If

$$\rho:\ vL \longrightarrow v^*K^*$$

is an embedding over vK and

$$\sigma:\ Lv \longrightarrow K^*v^*$$

is an embedding over Kv, then there exists an embedding

$$\iota:\ (L,v) \longrightarrow (K^*,v^*)$$

over K which respects ρ and σ.

Proof. We have already shown that every finitely generated subextension of $(L|K,v)$ embeds over (K,v) in (K^*,v^*) respecting both embeddings ρ and σ. Using the saturation property of (K^*,v^*) we have to deduce our assertion from this. To do so, we will work in an extended language $\mathcal{L}'$ consisting of the language $\mathcal{L}_{\mathrm{VF}}$ of valued fields together with the predicates

$$\begin{array}{ll} \mathcal{P}_\alpha\,, & \alpha\in\rho(vL) \\ \mathcal{Q}_\zeta\,, & \zeta\in\sigma(Lv) \end{array}$$

which are interpreted in (K^*,v^*) such that

$$\begin{array}{rcl} \mathcal{P}_\alpha(a) & \Longleftrightarrow & v^*a=\alpha \\ \mathcal{Q}_\zeta(a) & \Longleftrightarrow & av^*=\zeta \end{array}$$

for all $a\in K^*$ and in (L,v) such that

$$\begin{array}{rcl} \mathcal{P}_\alpha(b) & \Longleftrightarrow & \rho(vb)=\alpha \\ \mathcal{Q}_\zeta(b) & \Longleftrightarrow & \sigma(bv)=\zeta \end{array}$$

for all $b\in L$. Note that these interpretations coincide on K.

We show that (K^*,v^*) remains $|L|^+$-saturated in the extended language $\mathcal{L}'$. To this end, we choose a subset $S_v\subset K^*$ of representatives for all values α in $\rho(vL)$, and a subset $S_r\subset K^*$ of representatives for all residues ζ in $\sigma(Lv)$. We compute

$$\begin{array}{rcl} |S_v| & = & |\rho vL| = |vL| \leq |L| < |L|^+\,, \\ |S_r| & = & |\sigma Lv| = |Lv| \leq |L| < |L|^+\,, \end{array}$$

hence $|S_v\cup S_r| < |L|^+$. Consequently, it follows that (K^*,v^*) remains $|L|^+$-saturated in the extended language $\mathcal{L}_{\mathrm{VF}}(S_v\cup S_r)$ (the new constants are interpreted in K^* by the corresponding elements from $S_v\cup S_r$). Now the predicates $\mathcal{P}_\alpha$ and $\mathcal{Q}_\zeta$ become definable in the language $\mathcal{L}_{\mathrm{VF}}(S_v\cup S_r)$. Indeed, if $\alpha\in\rho(vL)$, then we choose $b_\alpha\in S_v$ such that $vb_\alpha=\alpha$ and define $\mathcal{P}_\alpha(x) :\Leftrightarrow vx=vb_\alpha$. If $\zeta\in\sigma(Lv)$, then we choose $b_\zeta\in S_r$ such that $b_\zeta v^*=\zeta$ and define $\mathcal{Q}_\zeta(x) :\Leftrightarrow v^*(x-b_\zeta)>0$. Since (K^*,v^*) is $|L|^+$-saturated in the language $\mathcal{L}_{\mathrm{VF}}(S_v\cup S_r)$, it follows that it is also $|L|^+$-saturated in the language $\mathcal{L}'(S_v\cup S_r)$ and thus also in the language $\mathcal{L}'$, as asserted.

An embedding ι of an arbitrary subextension (F,v) of $(L|K,v)$) in (K^*,v^*) over K respects the predicates $\mathcal{P}_\alpha$ and $\mathcal{Q}_\zeta$ if and only if it satisfies, for all $b\in L_0$,

$$\rho(vb)=\alpha \Longleftrightarrow (F,v)\models\mathcal{P}_\alpha(b) \Longleftrightarrow (K^*,v^*)\models\mathcal{P}_\alpha(\iota b) \Longleftrightarrow v^*(\iota b)=\alpha\,,$$
$$\sigma(bv)=\zeta \Longleftrightarrow (F,v)\models\mathcal{Q}_\zeta(b) \Longleftrightarrow (K^*,v^*)\models\mathcal{Q}_\zeta(\iota b) \Longleftrightarrow (\iota b)v^*=\zeta\,,$$

which expresses the property of ι to respect the embeddings ρ and σ. We know that for every finitely generated subextension of $(L|K,v)$ there exists such an embedding ι. The saturation property of (K^*,v^*) now yields an embedding of (L,v) in (K^*,v^*) over K which respects the predicates and thus the embeddings ρ and σ. This completes the proof of our lemma. □

5.3. **Completions.** In this section, we deal with extensions of a valued field within its completion. This is a preparation for the subsequent section on the model theory of separably tame fields. But the results are also of independent interest. As a preparation for the next theorem, we need:

Lemma 5.8. *Assume that $(K(x)|K,v)$ is an extension within the completion of (K,v) such that x is transcendental over K. Then x is the limit of a pseudo Cauchy sequence in (K,v) of transcendental type.*

Proof. Since $x\in K^c$, it is the limit of a Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v), that is, the values $v(x-a_\nu)$ are strictly increasing with ν and are cofinal in vK. Suppose that this sequence would not be of transcendental type. Then there is a polynomial $f\in K[X]$ of least degree for which the values $vf(a_\nu)$ are not ultimately fixed. By Lemma 8 of [Ka],

$$vf(a_\nu) \;=\; \beta_h \,+\, hv(x-a_\nu)$$

holds for all large enough ν, where $\beta_h\in vK$ and h is a power of p. By Lemma 9 of [Ka],

$$vf(x) \;>\; \beta_h \,+\, hv(x-a_\nu)$$

for all large enough ν. As these values are cofinal in vK, we conclude that $vf(x)=\infty$, that is, $f(x)=0$. Hence if x is transcendental over K, then $(a_\nu)_{\nu<\lambda}$ must be of transcendental type. □

Theorem 5.9. *Let (K,v) be a henselian field. Assume that $(L|K,v)$ is a separable subextension of $(K^c|K,v)$. Then (K,v) is existentially closed in (L,v). In particular, every henselian inseparably defectless field is existentially closed in its completion.*

Proof. By Lemma 5.2, it suffices to show that (K,v) is existentially closed in every subfield (F,v) of (L,v) which is finitely generated over K. Equivalently, it suffices to show that (K,v) is existentially closed in $(F,v)^h$; note that $(F,v)^h\subset(K,v)^c$ since the completion of a henselian field is again henselian (cf. [W], Theorem 32.19). As a subextension of the separable extension $L|K$, also $F|K$ is separable. So we may choose a separating transcendence basis $\mathcal{T}=\{x_1,\ldots,x_n\}$ of $F|K$. Then (F,v) lies in the completion of $(K(\mathcal{T}),v)$ since it lies in the completion of (K,v). The completion of $K(\mathcal{T})^h$ is equal to K^c since $K(\mathcal{T})\subseteq K^c$ and (K^c,v) is henselian. Consequently, F^h lies in the completion of $K(\mathcal{T})^h$. On the other hand, $F^h|K(\mathcal{T})^h$ is a finite separable extension; since a henselian field is separable-algebraically closed in its completion (cf. [W], Theorem 32.19), it must be trivial. That is,

$$(F,v)^h=(K(x_1,\ldots,x_n),v)^h\;.$$

Set $F_0=K$ and $(F_i,v)=(K(x_1,\ldots,x_i),v)^h$, $1\leq i\leq n$, where the henselization is taken within F^h. Now it suffices to show that $(F_{i-1},v)\prec_\exists(F_i,v)$ for $1\leq i\leq n$. As x_i is an element of the completion K^c of (F_{i-1},v), it is the limit of a Cauchy sequence in (F_{i-1},v). Since x_i is transcendental over F_{i-1}, this Cauchy sequence must be of transcendental type

by Lemma 5.8. Hence by Corollary 6.3, $(F_{i-1},v) \prec_\exists (F_{i-1}(x_i),v)^h$ for $1 \leq i \leq n$, which in view of $(F_{i-1}(x_i),v)^h = (F_i,v)^h$ proves our assertion.

The second assertion of our theorem follows from the first and the fact that if (K,v) is inseparably defectless, then the immediate extension $K^c|K$ is separable, according to Corollary 2.8. □

From this theorem together with part b) of Lemma 5.3, we obtain:

Corollary 5.10. *A henselian field (K,v) is existentially closed in its completion K^c if and only if the extension $K^c|K$ is separable.*

This leads to the following question:

Open Problem: Take any field k. Which are the subfields $K \subset k((t))$ with $t \in K$ such that $k((t))|K$ is separable?

Recall that v_t denotes the t-adic valuation on $k(t)$ and on $k((t))$. Since $(k((t)),v_t)$ is henselian, we can choose the henselization $(k(t),v_t)^h$ in $(k((t)),v_t)$. Then $(k((t)),v_t)$ is the completion of both $(k(t),v_t)$ and $(k(t),v_t)^h$. Further, (k,v_t) is trivially valued and thus defectless. By Theorem 1.7, it follows that $(k(t),v_t)^h$ is henselian and defectless. Now Corollary 2.8 shows:

Corollary 5.11. *The extension $k((t))|k(t)^h$ is regular.*

Using Theorem 5.9, we conclude:

Theorem 5.12. *Let k be an arbitrary field. Then $(k(t),v_t)^h \prec_\exists (k((t)),v_t)$.*

This result also follows from Theorem 2 of [Er6]. It is used in [K6] in connection with the characterization of large fields.

To give a further application, we need another lemma.

Lemma 5.13. *Let t be transcendental over K. Suppose that K admits a nontrivial henselian valuation v. Then $(K,v) \prec_\exists (K(t),v_t\circ v)^h$.*

Proof. Let (K^*,v^*) be a $|K(t)^h|^+$-saturated elementary extension of (K,v). Then by Corollary 4.3, v^*K^* is a $|vK|^+$-saturated elementary extension of vK. Hence, there exists an element $\alpha \in v^*K^*$ such that $\alpha > vK$. We also have that $(v_t\circ v)t > vK$. Now if $\Gamma \subset \Delta$ is an extension of ordered abelian groups and $\Delta \ni \alpha > \Gamma$, then the ordering on $\mathbb{Z}\alpha + \Gamma$ is uniquely determined. Indeed, $\mathbb{Z}\alpha+\Gamma$ is isomorphic to the product $\mathbb{Z}\alpha \amalg \Gamma$, lexicographically ordered. So we see that the assignment $(v_t\circ v)t \mapsto \alpha$ induces an embedding of $(v_t\circ v)K(t) \simeq \mathbb{Z}(v_t\circ v)t \times vK$ (with the lexicographic ordering) in v^*K^* over vK as ordered groups. Now choose $t^* \in K^*$ such that $v^*t^* = \alpha$. As $(v_t\circ v)t$ and α are not torsion elements over vK, Lemma 2.2 shows that the assignment $t \mapsto t^*$ induces an embedding of $(K(t),v_t\circ v)$ in (K^*,v^*) over K. Since (K,v) is henselian, so is the elementary extension (K^*,v^*). By the universal property of the henselization, the embedding can thus be extended to an embedding of $(K(t),v_t\circ v)^h$ in (K^*,v^*). By Proposition 5.1, this gives our assertion. □

Now we are able to prove:

Theorem 5.14. *If the field K admits a nontrivial henselian valuation, then $K \prec_\exists K((t))$ (as fields).*

Proof. Let v be the nontrivial valuation on K for which (K,v) is henselian. By Lemma 5.13, we have that $(K,v) \prec_\exists (K(t), v_t\circ v)^h$. By Corollary 5.11, $K((t))|K(t)^h$ is separable. Since $(K((t)),v_t)$ is the completion of $(K(t),v_t)$, it follows that $(K((t)),v_t\circ v)$ is the completion of $(K(t),v_t\circ v)$. Hence, Theorem 5.9 shows that $(K(t),v_t\circ v)^h \prec_\exists (K((t)),v_t\circ v)$. It follows that $(K,v)\prec_\exists (K((t)),v_t\circ v)$. In particular, $K\prec_\exists K((t))$, as asserted. □

6. The Relative Embedding Property

Inspired by the assertion of Lemma 5.7, we define a property that will play a key role in our approach to the model theory of tame fields. Let $\mathbf{C}$ be a class of valued fields. We will say that $\mathbf{C}$ has the **Relative Embedding Property**, if the following holds:

if $(L,v),(K^*,v^*)\in\mathbf{C}$ with common subfield (K,v) such that
- (K,v) is defectless,
- (K^*,v^*) is $|L|^+$-saturated,
- vL/vK is torsion free and $Lv|Kv$ is separable,
- there are embeddings $\rho:\ vL\longrightarrow v^*K^*$ over vK and $\sigma:\ Lv\longrightarrow K^*v^*$ over Kv,

then there exists an embedding $\iota:\ (L,v)\longrightarrow (K^*,v^*)$ over K which respects ρ and σ.

We will show that the Relative Embedding Property of $\mathbf{C}$ implies another property of $\mathbf{C}$ which is very important for our purposes. If $\mathfrak{C}\subset\mathfrak{A}$ and $\mathfrak{C}\subset\mathfrak{B}$ are extensions of $\mathcal{L}$-structures, then we will write $\mathfrak{A}\equiv_{\mathfrak{C}}\mathfrak{B}$ if $(\mathfrak{A},\mathfrak{C})\equiv(\mathfrak{B},\mathfrak{C})$ in the language $\mathcal{L}(\mathfrak{C})$ augmented by constant names for the elements of $\mathfrak{C}$. If for every two fields $(L,v),(F,v)\in\mathbf{C}$ and every common defectless subfield (K,v) of (L,v) and (F,v) such that vL/vK is torsion free and $Lv|Kv$ is separable, the side conditions $vL\equiv_{vK} vF$ and $Lv\equiv_{Kv} Fv$ imply that $(L,v)\equiv_{(K,v)}(F,v)$, then we will call $\mathbf{C}$ **relatively subcomplete**. Note that if $\mathbf{C}$ is a relatively subcomplete class of defectless fields, then $\mathbf{C}$ is relatively model complete because by Lemma 5.3, the side conditions $vK\prec vL$ and $Kv\prec Lv$ imply that $vK\equiv_{vK} vL$ and $Kv\equiv_{Kv} Lv$. But relative model completeness is weaker than relative subcompleteness, because $vL\equiv_{vK} vF$ does not imply that $vK\prec vL$, and $Lv\equiv_{Kv} Fv$ does not imply that $Kv\prec Lv$.

The following lemma shows that the Relative Embedding Property is a powerful property:

Lemma 6.1. *Take an elementary class $\mathbf{C}$ of defectless valued fields which has the Relative Embedding Property. Then $\mathbf{C}$ is relatively subcomplete and relatively model complete, and the AKE$^\exists$ Principle is satisfied by all extensions $(L|K,v)$ such that both $(K,v),(L,v)\in\mathbf{C}$. If moreover all fields in $\mathbf{C}$ are of fixed equal characteristic, then $\mathbf{C}$ is relatively complete.*

Proof. Let us first show that $(L|K,v)$ satisfies the AKE$^\exists$ Principle whenever $(K,v),(L,v)\in\mathbf{C}$. So assume that $vK\prec_\exists vL$ and $Kv\prec_\exists Lv$. We take an $|L|^+$-saturated elementary extension (K^*,v^*) of (K,v). Since $\mathbf{C}$ is assumed to be an elementary class, $(K,v)\in\mathbf{C}$ implies that $(K^*,v^*)\in\mathbf{C}$. Because of $vK\prec_\exists vL$ and $Kv\prec_\exists Lv$, there are embeddings $vL\to v^*K^*$ over vK and $Lv\to K^*v^*$ over Kv by Proposition 5.1. Moreover, vL/vK is torsion free and $Lv|Kv$ is separable by Lemma 5.3. So by the Relative Embedding Property there is an embedding of (L,v) in (K^*,v^*) over K, which shows that $(K,v)\prec_\exists (L,v)$.

Now assume that $(L,v),(F,v)\in\mathbf{C}$ with common defectless subfield (K,v) such that vL/vK is torsion free, $Lv|Kv$ is separable, $vL\equiv_{vK} vF$ and $Lv\equiv_{Kv} Fv$. We have to show that $(L,v)\equiv_{(K,v)}(F,v)$.

To begin with, we construct an elementary extension (L_0,v) of (L,v) and an elementary extension (F_0,v) of (F,v) such that $vL_0=vF_0$ and $L_0v=F_0v$. Our condition $vL\equiv_{vK} vF$ means that vL and vF are equivalent in the augmented language $\mathcal{L}_{\rm OG}(vK)$ of ordered groups with constants from vK. Similarly, $Lv\equiv_{Kv} Fv$ means that Lv and Fv are equivalent in the augmented language $\mathcal{L}_{\rm R}$ of rings with constants from Kv. It follows from the proof of Theorem 6.1.15 in [C–K] that we can choose a cardinal λ and an ultrafilter $\mathcal{D}$ on λ such that $\prod_\lambda vL/\mathcal{D}\simeq\prod_\lambda vF/\mathcal{D}$ and $\prod_\lambda Lv/\mathcal{D}\simeq\prod_\lambda Fv/\mathcal{D}$ in the respective augmented languages. But this means that for $(L_0,v):=\prod_\lambda(L,v)/\mathcal{D}$ and $(F_0,v):=\prod_\lambda(F,v)/\mathcal{D}$, we have that $vL_0=\prod_\lambda vL/\mathcal{D}$ is isomorphic over vK to $vF_0=\prod_\lambda vF/\mathcal{D}$, and $L_0v=\prod_\lambda Lv/\mathcal{D}$ is isomorphic over Kv to $F_0v=\prod_\lambda vF/\mathcal{D}$. Passing to an equivalent valuation on L_0 which still extends the valuation v of K, we may assume that $vL_0=vF_0$; similarly, passing to an equivalent residue map we may assume that $L_0v=F_0v$. As vL/vK and vF/vK are torsion free by assumption and vL_0/vL and vF_0/vF are torsion free since $vL\prec vL_0$ and $vF\prec vF_0$, we find that vL_0/vK and vF_0/vK are torsion free. Similarly, one shows that $L_0v|Kv$ and $F_0v|Kv$ are separable.

Now we construct two elementary chains $((L_i,v))_{i<\omega}$ and $((F_i,v))_{i<\omega}$ as follows. We choose a cardinal $\kappa_0=\max\{|L_0|,|F_0|\}$. By induction, for every $i<\omega$ we take (L_{i+1},v) to be a κ_i^+-saturated elementary extension of (L_i,v), where $\kappa_i=\max\{|L_i|,|F_i|\}$, and (F_{i+1},v) to be a κ_i^+-saturated elementary extension of (F_i,v). We can take $(L_{i+1},v)=\prod_{\lambda_i}(L_i,v)/\mathcal{D}_i$ and $(F_{i+1},v)=\prod_{\lambda_i}(F_i,v)/\mathcal{D}_i$ for suitable cardinals λ_i and ultrafilters $\mathcal{D}_i$; this yields that $vL_i=vF_i$ and $L_iv=F_iv$ for all i.

All (L_i,v) and (F_i,v) are elementary extensions of (L,v) and (F,v) respectively, so it follows that they lie in $\mathbf{C}$ and in particular, are defectless fields. We take (L^*,v) to be the union over the elementary chain (L_i,v), $i<\omega$; so $(L,v)\prec(L^*,v)$. Similarly, we take (F^*,v) to be the union over the elementary chain (F_i,v), $i<\omega$; so $(F,v)\prec(F^*,v)$. Now we carry out a back and forth construction that will show that (L^*,v) and (F^*,v) are isomorphic over K.

We start by embedding (L_0,v) in (F_1,v). The identity mappings are embeddings of vL_0 in vF_1 over vK and of L_0v in F_1v over Kv, and we know that vL_0/vK is torsion free and $L_0v|Kv$ is separable. Since (F_1,v) is κ_0^+-saturated with $\kappa_0\geq|L_0|$, and since (K,v) is defectless, we can apply the Relative Embedding Property to find an embedding ι_0 of (L_0,v) in (F_1,v) over K which respects the embeddings of the value group and the residue field. That is, we have that $v\iota_0L_0=vF_0$ and $(\iota_0L_0)v=F_0v$.

The isomorphism $\iota_0^{-1}:\ \iota_0L_0\rightarrow L_0$ can be extended to an isomorphism ι_0^{-1} from F_1 onto an extension field of L_0 which we will simply denote by $\iota_0^{-1}F_1$. We take the valuation on this field to be the one induced via ι_0^{-1} by the valuation on F_1. Hence, ι_0^{-1} induces an isomorphism on the value groups and the residue fields, so that we obtain that $v\iota_0^{-1}F_1=vF_1=vL_1$ and $(\iota_0^{-1}F_1)v=F_1v=L_1v$. The identity mappings are embeddings of $v\iota_0^{-1}F_1$ in vL_2 over vL_0 and of $(\iota_0^{-1}F_1)v$ in L_2v over L_0v. Since $vL_0\prec vL_1$ and $L_0v\prec L_1v$, we know that $v\iota_0^{-1}F_1/vL_0$ is torsion free and $(\iota_0^{-1}F_1)v|L_0v$ is separable. Since (L_2,v) is κ_1^+-saturated with $\kappa_1\geq|F_1|=|\iota_0^{-1}F_1|$, and since (L_0,v) is defectless, we can apply

the Relative Embedding Property to find an embedding $\tilde{\iota}_1$ of $(\iota_0^{-1}F_1, v)$ in (L_2, v) over L_0 which respects the embeddings of the value group and the residue field. That is, we obtain an embedding $\iota'_1 := \tilde{\iota}_1\iota^{-1}$ of F_1 in L_2 over K. We note that $L_0 \subset \iota'_1 F_1$ and that ${\iota'_1}^{-1} : \iota'_1 F_1 \to F_1$ extends ι_0.

Suppose that we have constructed, for an even i, the embeddings

$$\begin{array}{rl} \iota_i : & (L_i, v) \longrightarrow (F_{i+1}, v) \\ \iota'_{i+1} : & (F_{i+1}, v) \longrightarrow (L_{i+2}, v) \end{array}$$

as embeddings over K, such that $L_i \subset \iota'_{i+1}F_{i+1}$ and that ${\iota'_{i+1}}^{-1} : \iota'_1 F_{i+1} \to F_{i+1}$ extends ι_i. We wish to construct similar embeddings for $i+2$ in place of i.

The isomorphism ${\iota'_{i+1}}^{-1} : \iota'_1 F_{i+1} \to F_{i+1}$ can be extended to an isomorphism ${\iota'_{i+1}}^{-1}$ from L_{i+2} onto an extension field of F_{i+1} which we will denote by ${\iota'_{i+1}}^{-1}L_{i+2}$; this isomorphism extends ι_i. We take the valuation on this field to be the one induced via ${\iota'_{i+1}}^{-1}$ by the valuation on L_{i+2}. We obtain that $v{\iota'_{i+1}}^{-1}L_{i+2} = vL_{i+2} = vF_{i+2}$ and $({\iota'_{i+1}}^{-1}L_{i+2})v = L_{i+2}v = F_{i+2}v$. The identity mappings are embeddings of $v{\iota'_{i+1}}^{-1}L_{i+2}$ in vF_{i+3} over vF_{i+1} and of $({\iota'_{i+1}}^{-1}L_{i+2})v$ in $F_{i+3}v$ over $F_{i+1}v$. Since $vF_{i+1} \prec vF_{i+3}$ and $F_{i+1}v \prec F_{i+3}v$, we know that $v{\iota'_{i+1}}^{-1}L_{i+2}/vF_{i+1}$ is torsion free and $({\iota'_{i+1}}^{-1}L_{i+2})v|F_{i+1}v$ is separable. Since (F_{i+3}, v) is κ_{i+2}^+-saturated with $\kappa_{i+2} \geq |L_{i+2}| = |{\iota'_{i+1}}^{-1}L_{i+2}|$, and since (F_{i+1}, v) is defectless, we can apply the Relative Embedding Property to find an embedding $\tilde{\iota}'_{i+2}$ of $({\iota'_{i+1}}^{-1}L_{i+2}, v)$ in (F_{i+3}, v) over F_{i+1} which respects the embeddings of the value group and the residue field. We obtain an embedding $\iota_{i+2} := \tilde{\iota}'_{i+2}{\iota'_{i+1}}^{-1}$ of L_{i+2} in F_{i+3}; since $\tilde{\iota}'_{i+2}$ is the identity on $\iota_i L_i \subset F_{i+1}$, this embedding extends ι_i. We note that $F_{i+1} \subset \iota_{i+2}L_{i+2}$ and that $\iota_{i+2}^{-1} : \iota_{i+2}L_{i+2} \to L_{i+2}$ extends ι'_{i+1}.

From now on, the construction of ι'_{i+2} is similar to that of ι'_i for every odd $i \geq 1$, and the construction of ι_{i+2} is similar to that of ι'_i for every even $i \geq 2$.

Now we take ι to be the set theoretical union over the embeddings ι_i, $i < \omega$ even. Then ι is an embedding of (L^*, v) in (F^*, v). It is onto since F_i lies in the image of ι_{i+1}, for every odd i. So we have obtained an isomorphism from (L^*, v) onto (F^*, v) over K, which shows that $(L^*, v) \equiv_{(K,v)} (F^*, v)$. Since $(L, v) \prec (L^*, v)$ and $(F, v) \prec (F^*, v)$, this implies that $(L, v) \equiv_{(K,v)} (F, v)$, as required. We have proved that $\mathbf{C}$ is relatively subcomplete, and we know already that this implies that $\mathbf{C}$ is relatively model complete.

Finally, assume in addition that all fields in $\mathbf{C}$ are of fixed equal characteristic. We wish to show that $\mathbf{C}$ is relatively complete. So take $(L, v), (F, v) \in \mathbf{C}$ such that $vL \equiv vF$ and $Lv \equiv Fv$. Fixed characteristic means that L and F have a common prime field K. The assumption that both (L, v) and (F, v) are of equal characteristic means that the restrictions of their valuations to K is trivial. Hence, $vK = 0$ and consequently, vL/vK is torsion free and $vL \equiv vF$ implies that $vL \equiv_{vK} vF$. Further, $K = Kv$ is also the prime field of Lv and Fv, so $Lv \equiv Fv$ implies that $Lv \equiv_{Kv} Fv$. Since a prime field is always perfect, we also have that $Lv|Kv$ is separable. As a trivially valued field, (K, v) is defectless. From what we have already proved, we obtain that $(L, v) \equiv_{(K,v)} (F, v)$, which implies that $(L, v) \equiv (F, v)$. $\square$

Now we look for a criterion for an elementary class of valued fields to have the Relative Embedding Property. Somehow, we have to improve Embedding Lemma II (Lemma 5.7) to cover the case of extensions $(L|K,v)$ with transcendence defect. Loosely speaking, these contain an immediate part. The idea is to require that this part can be treated separately, that is, that we find an intermediate field $(L',v)\in\mathbf{C}$ such that $(L|L',v)$ is immediate and $(L'|K,v)$ has no transcendence defect. The immediate part has then to be handled by a new approach which we will describe in the following embedding lemma. Note that by Theorem 1 of [Ka] together with Theorem 2.14, the hypothesis on x does automatically hold if (K,v) is algebraically maximal.

Lemma 6.2. (Embedding Lemma III)
Let $(K(x)|K,v)$ be a nontrivial immediate extension of valued fields. If x is the limit of a pseudo Cauchy sequence of transcendental type in (K,v), then $(K(x),v)^h$ embeds over K in every $|K|^+$-saturated henselian extension $(K,v)^*$ of (K,v).

Proof. Take a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ of transcendental type in (K,v) with limit x. Then the collection of elementary formulas "$v(x-a_\nu)=v(a_{\nu+1}-a_\nu)$", $\nu<\lambda$, is a (partial) type over (K,v). Since $(K,v)^*$ is $|K|^+$-saturated, there is an element $x^*\in K^*$ such that $v^*(x^*-a_\nu)=v^*(a_{\nu+1}-a_\nu)$ holds for all $\nu<\lambda$. That is, x^* is also a limit of $(a_\nu)_{\nu<\lambda}$. By Theorem 2.15, the homomorphism induced by $x\mapsto x^*$ is an embedding of $(K(x),v)$ over K in $(K,v)^*$. By the universal property of the henselization, this embedding can be extended to an embedding of $(K(x),v)^h$ over K in $(K,v)^*$, since the latter is henselian by hypothesis. $\square$

Note that the lemma fails if the condition on the pseudo Cauchy sequence to be transcendental is omitted, even if we require in addition that (K,v) is henselian. There may exist nontrivial finite immediate extensions $(K(x)|K,v)$ of henselian fields; for a comprehensive collection of examples, see [K9]. On the other hand, K^* may be a regular extension of K (e.g., this is always the case if $(K,v)^*$ is an elementary extension of (K,v)), and then, $K(x)$ does certainly not admit an embedding over K in K^*.

The model theoretic application of Embedding Lemma III is:

Corollary 6.3. *Let (K,v) be a henselian field and $(K(x)|K,v)$ an immediate extension such that x is the limit of a pseudo Cauchy sequence of transcendental type in (K,v). Then $(K,v)\prec_\exists (K(x),v)^h$. In particular, an algebraically maximal field is existentially closed in every henselization of an immediate rational function field of transcendence degree* 1.

Proof. Choose $(K,v)^*$ to be a $|K|^+$-saturated elementary extension of (K,v). Since "henselian" is an elementary property, $(K,v)^*$ will also be henselian. Apply Embedding Lemma III and Proposition 5.1. $\square$

Now we are able to give the announced criterion:

Lemma 6.4. *Let $\mathbf{C}$ be an elementary class of valued fields which satisfies*

(CALM) *every field in $\mathbf{C}$ is algebraically maximal,*

(CRAC) *if $(L,v)\in\mathbf{C}$ and K is relatively algebraically closed in L such that $Lv|Kv$ is algebraic and vL/vK is a torsion group, then $(K,v)\in\mathbf{C}$ with $Lv=Kv$ and $vL=vK$,*

(CIMM) *if $(K,v) \in \mathbf{C}$, then every henselization of an immediate function field of transcendence degree* 1 *over (K,v) is already the henselization of a rational function field.*

Then $\mathbf{C}$ *has the Relative Embedding Property.*

Proof. Assume that the elementary class $\mathbf{C}$ satisfies (CALM), (CRAC) and (CIMM). Take $(L,v),(K^*,v^*) \in \mathbf{C}$ with (K^*,v^*) being $|L|^+$-saturated, a valued subfield (K,v) of (L,v) and (K^*,v^*) such that vL/vK is torsion free and $Lv|Kv$ is separable, and embeddings $\rho :\ vL \to v^*K^*$ over vK and $\sigma :\ Lv \to K^*v^*$ over Kv. We have to show that there exists an embedding $\iota :\ (L,v) \to (K^*,v^*)$ over K which respects ρ and σ.

Take the set $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\}$ as in the proof of Corollary 3.17. Then $vL/vK(\mathcal{T})$ is a torsion group and $Lv|K(\mathcal{T})v$ is algebraic. Let K' be the relative algebraic closure of $K(\mathcal{T})$ within L. It follows that also vL/vK' is a torsion group and $Lv|K'v$ is algebraic. Hence by condition (CRAC), we have that $(K',v) \in \mathbf{C}$ with $Lv = K'v$ and $vL = vK'$, which shows that the extension $L|K'$ is immediate. On the other hand, $\mathcal{T}$ is a standard valuation transcendence basis of $(K'|K,v)$ by construction, hence according to Corollary 2.4, this extension has no transcendence defect. Since (K,v) is defectless by assumption and (K^*,v^*) is henselian by condition (CALM), Lemma 5.7 gives an embedding of (K',v) in (K^*,v^*) over K which respects ρ and σ. Now we have to look for an extension of this embedding to (L,v). Since $(L|K',v)$ is immediate, such an extension will automatically respect ρ and σ.

We identify K' with its image in K^*. In view of Lemma 5.2, it remains to show that every finitely generated subextension (F,v) of $(L|K',v)$ embeds over K' in (K^*,v^*). We apply our slicing approach. Since F is finitely generated over K', it has a finite transcendence basis $\{t_1,\ldots,t_n\}$ over K'. Let us put $K_0 = K'$ and K_i to be the relative algebraic closure of $K(t_1,\ldots,t_i)$ in L for $1 \le i \le n$. Then K_n contains F, and by condition (CRAC), every (K_i,v) is a member of $\mathbf{C}$. Moreover, $\mathrm{trdeg}(K_{i+1}|K_i) = 1$ for $0 \le i < n$. We proceed by induction on i. If we have shown that (K_i,v) embeds in (K^*,v^*) over K', then we identify it with its image. Hence it now remains to show that the immediate extension (K_{i+1},v) of transcendence degree 1 embeds in (K^*,v^*) over K_i. Since (K^*,v^*) is $|L|^+$-saturated, it is also $|K_{i+1}|^+$-saturated. Hence again, Lemma 5.2 shows that it suffices to prove the existence of an embedding for every finitely generated subextension (F_{i+1},v) of $(K_{i+1}|K_i,v)$. The henselization of $(F_{i+1}|K_i,v)$ is an immediate function field of transcendence degree 1, so by condition (CIMM), its henselization is the henselization of a rational function field. Since (K_i,v) is algebraically maximal by condition (CALM), Embedding Lemma III (Lemma 6.2) now yields that there is an embedding of (F_{i+1},v) in (K^*,v^*) over K_i. This completes our proof by induction. □

7. The model theory of tame and separably tame fields

7.1. **Tame fields.** We have already shown in Lemma 3.10 that in positive characteristic, the class of tame fields coincides with the class of algebraically maximal perfect fields. Let us show that the property of being a tame field of fixed residue characteristic is elementary. If the residue characteristic is fixed to be 0 then by Lemma 3.5, "tame" is equivalent to "henselian" which is axiomatized by the axiom scheme (HENS). Now assume that the residue characteristic is fixed to be a positive prime p. By Lemma 3.10, a valued

field of positive residue characteristic is tame if and only if it is an algebraically maximal field having p-divisible value group and perfect residue field. A valued field (K,v) has p-divisible value group if and only if it satisfies the following elementary axiom:

(VGD$_p$) $\quad \forall x\,\exists y:\ vxy^p = 0$.

Furthermore, (K,v) has perfect residue field if and only if it satisfies:

(RFD$_p$) $\quad \forall x\,\exists y:\ vx = 0 \to v(xy^p - 1) > 0$.

Finally, the property of being algebraically maximal is axiomatized by the axiom schemes (HENS) and (MAXP). We summarize: The **theory of tame fields of residue characteristic** 0 is just the theory of henselian fields of residue characteristic 0. If p is a prime, then the **theory of tame fields of residue characteristic** p is the theory of valued fields together with axioms (VGD$_p$), (RFD$_p$), (HENS) and (MAXP). Now we also see how to axiomatize the theory of all tame fields. Indeed, for residue characteristic 0 there are no conditions on the value group and the residue field. For residue characteristic $p > 0$, we have to require (VGD$_p$) and (RFD$_p$). We can do this by the axiom scheme

(TAD) $\quad v(\underbrace{1+\ldots+1}_{p\ \text{times}}) > 0 \to (\mathrm{VGD}_p) \wedge (\mathrm{RFD}_p) \qquad (p \text{ prime})$.

So the **theory of tame fields** is the theory of valued fields together with axioms (TAD), (HENS) and (MAXP).

Recall that by part a) of Corollary 3.12, a valued field of positive characteristic is tame if and only if it is algebraically maximal and perfect. We have already seen that every AKE$^\exists$-field must be henselian and defectless and in particular, algebraically maximal. Therefore, the model theory of tame fields that we will develop now is representative of the model theory of perfect valued fields in positive characteristic.

Let $\mathbf{C}$ be the elementary class of all tame fields. By Lemma 3.6, all tame fields are henselian defectless, so $\mathbf{C}$ satisfies condition (CALM) of Lemma 6.4. By Lemma 3.15, it also satisfies condition (CRAC). Finally, it satisfies (CIMM) by virtue of Theorem 1.9. Hence, we can infer from Lemma 6.4 and Lemma 6.1:

Theorem 7.1. *The elementary class of tame fields has the Relative Embedding Property and is relatively subcomplete and relatively model complete. Every elementary class of tame fields of fixed equal characteristic is relatively complete.*

Lemma 6.4 does not give the full information about the AKE$^\exists$ Principle because it requires that not only (K,v), but also (L,v) is a member of the class $\mathbf{C}$. If the latter is not the case, then it just suffices if one can show that it is contained in a member of $\mathbf{C}$. To this end, we need the following lemma:

Lemma 7.2. *If Γ is a p-divisible ordered abelian group and $\Gamma \prec_\exists \Delta$, then Γ is also existentially closed in the p-divisible hull of Δ. If k is a perfect field and $k \prec_\exists \ell$, then k is also existentially closed in the perfect hull of ℓ.*

If (K,v) is a tame field and $(L|K,v)$ an extension with $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$, then every maximal purely wild extension (W,v) of (L,v) is a tame field satisfying $vK \prec_\exists vW$ and $Kv \prec_\exists Wv$.

Proof. By Proposition 5.1, $\Gamma \prec_\exists \Delta$ implies that Δ embeds over Γ in every $|\Delta|^+$-saturated elementary extension of Γ. Such an elementary extension is p-divisible like Γ. Hence, the

embedding can be extended to an embedding of $\frac{1}{p^\infty}\Delta$, which by Proposition 5.1 shows that $\Gamma \prec_\exists \frac{1}{p^\infty}\Delta$.

Again by the same lemma, $k \prec_\exists \ell$ implies that ℓ embeds over k in every $|\ell|^+$-saturated elementary extension of k. Such an elementary extension is perfect like k. Hence, the embedding can be extended to an embedding of ℓ^{1/p^∞}, which by Proposition 5.1 shows that $k \prec_\exists \ell^{1/p^\infty}$.

Now suppose that the assumptions of the final assertion of our lemma hold. By Corollary 3.13, (W,v) is a tame field. By Theorem 3.7, vW is the p-divisible hull $\frac{1}{p^\infty}vL$ of vL, and Wv is the perfect hull Lv^{1/p^∞} of Lv. So our assertion follows since we have just proved that vK (which is p-divisible by Lemma 3.10) is existentially closed in $\frac{1}{p^\infty}vL$ and that Kv (which is perfect by Lemma 3.10) is existentially closed in the perfect hull Lv^{1/p^∞} of Lv. □

Assume that (K,v) is a tame field and $(L|K,v)$ an extension such that $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$. We choose some maximal purely wild extension (W,v) of (L,v). According to the foregoing lemma, (W,v) is a tame field with $vK \prec_\exists vW$ and $Kv \prec_\exists Wv$. Hence by Theorem 7.1 together with Lemma 6.1, $(K,v) \prec_\exists (W,v)$. It follows that $(K,v) \prec_\exists (L,v)$. This proves the first assertion of Theorem 1.3.

Now let $\mathbf{C}$ be an elementary class of valued fields. We define

$$v\mathbf{C} := \{vK \mid (K,v) \in \mathbf{C}\} \quad \text{and} \quad \mathbf{C}v := \{Kv \mid (K,v) \in \mathbf{C}\}\ .$$

If both $v\mathbf{C}$ and $\mathbf{C}v$ are model complete elementary classes, then the side conditions $vK \prec vL$ and $Kv \prec Lv$ will hold for every two members $(K,v) \subset (L,v)$ of $\mathbf{C}$. Similarly, if $v\mathbf{C}$ and $\mathbf{C}v$ are complete elementary classes, then the side conditions $vK \equiv vL$ and $Kv \equiv Lv$ will hold for all $(K,v),(L,v) \in \mathbf{C}$. So we obtain from the foregoing theorems:

Theorem 7.3. *If $\mathbf{C}$ is an elementary class consisting of tame fields and if $v\mathbf{C}$ and $\mathbf{C}v$ are elementary model complete classes, then $\mathbf{C}$ is model complete. If $\mathbf{C}$ is an elementary class consisting of tame fields of fixed equal characteristic, and if $v\mathbf{C}$ and $\mathbf{C}v$ are elementary complete classes, then $\mathbf{C}$ is complete.*

Note that the converses are true by virtue of Corollary 4.2, provided that $v\mathbf{C}$ and $\mathbf{C}v$ are elementary classes.

Corollary 7.4. *Let $\mathbf{T}$ be an elementary theory consisting of all perfect valued fields of equal characteristic whose value groups satisfy a given model complete elementary theory $\mathbf{T}_{\rm vg}$ of ordered abelian groups and whose residue fields satisfy a given model complete elementary theory $\mathbf{T}_{\rm rf}$ of fields. Then the theory $\mathbf{T}^*$ of algebraically maximal valued fields satisfying $\mathbf{T}$ is the model companion of $\mathbf{T}$.*

Proof. It follows from Theorem 7.3 that $\mathbf{T}^*$ is model complete. For every model K of $\mathbf{T}$, any maximal immediate algebraic extension is a model of $\mathbf{T}^*$ (by Lemma 3.10); note that it is an extension of K having the same value group and residue field. □

In the case of positive characteristic, $\mathbf{T}^*$ is in general not a model completion since there exist perfect valued fields of positive characteristic which admit two nonisomorphic maximal immediate algebraic extensions, both being models of the model companion. In the

case of equal characteristic 0, the algebraically maximal fields are just the henselian fields, and we find that $\mathbf{T}^*$ is a model completion of $\mathbf{T}$, because henselizations are unique up to isomorphism.

A **weak prime model** in an elementary class $\mathbf{C}$ is a model in $\mathbf{C}$ that can be embedded in every other highly enough saturated member of $\mathbf{C}$. Elementary classes of tame fields of equal characteristic admit weak prime models if the elementary classes of their value groups and their residue fields do:

Theorem 7.5. *Let $\mathbf{C}$ be an elementary class consisting of tame fields of equal characteristic. Suppose that there exists an infinite cardinal κ, an ordered group Γ and a field k, both of cardinality $\leq \kappa$, such that Γ admits an elementary embedding in every κ^+-saturated member of $v\mathbf{C}$ and k admits an elementary embedding in every κ^+-saturated member of $\mathbf{C}v$. Then there exists $(K_0, v) \in \mathbf{C}$ of cardinality $\leq \kappa$, having value group Γ and residue field k, such that (K_0, v) admits an elementary embedding in every κ^+-saturated member of $\mathbf{C}$. Moreover, we can assume that (K_0, v) admits a standard valuation transcendence basis over its prime field.*

Proof. Take any $(E, v) \in \mathbf{C}$ and let $(E, v)^*$ be a κ^+-saturated elementary extension of (E, v). Then also v^*E^* and E^*v^* are κ^+-saturated. Since $\mathbf{C}$ is an elementary class, we find that $(E, v)^* \in \mathbf{C}$. Consequently, $(E, v)^*$ is a tame field. By Lemma 3.10, its value group is p-divisible and its residue field is perfect. By assumption, Γ admits an elementary embedding in v^*E^*, and k admits an elementary embedding in E^*v^*. Hence, also Γ is p-divisible and k is perfect.

Now by Lemma 3.14, there exists a tame field (K_0, v) of cardinality at most κ having value group Γ and residue field k and admitting a standard valuation transcendence basis over its prime field. If (K^*, v^*) is a κ^+-saturated model of $\mathbf{C}$, then v^*K^* and K^*v^* are κ^+-saturated models of $v\mathbf{C}$ and $\mathbf{C}v$ respectively. Hence by hypothesis, there exists an elementary embedding of Γ in v^*K^* over the trivial group $\{0\}$, and an elementary embedding of k in K^*v^* over the prime field k_0 of k. Now k_0 is at the same time the prime field of K_0v and of K^*v^*. As we are dealing with valued fields of equal characteristic, k_0 is also the prime field of K_0 and K^*, and the valuation v is trivial on k_0. We have that vK_0/vk_0 is torsion free and $K_0v|k_0v$ is separable. Now Embedding Lemma II (Lemma 5.7) shows the existence of an embedding of (K_0, v) in (K^*, v^*) over k_0. By virtue of Theorem 7.1, this embedding is elementary (because the embeddings of value group and residue field are). This shows that (K_0, v) is elementarily embeddable in every κ^+-saturated model of $\mathbf{C}$. This in turn shows that (K_0, v) is a model of $\mathbf{C}$ and thus a weak prime model of $\mathbf{C}$. $\square$

The weak prime models that we have constructed in the foregoing proof have the special property that they admit a standard valuation transcendence basis over their prime field. The following corollary confirms the representative role of models with this property.

Corollary 7.6. *For every tame field (L, v) of arbitrary characteristic, there exists a subfield $(K, v) \prec (L, v)$ such that (K, v) admits a standard valuation transcendence basis over its prime field and $(L|K, v)$ is immediate.*

Proof. According to Lemma 3.17, for every tame field (L,v) there exists a subfield (K,v) of (L,v) admitting a standard valuation transcendence basis over its prime field, such that $(L|K,v)$ is immediate. In view of Theorem 7.1, the latter fact shows that $(K,v) \prec (L,v)$. □

As a final example, we consider the theory of tame fields of fixed positive characteristic with divisible or p-divisible value groups and fixed finite residue field.

Theorem 7.7. *a) Every elementary class* $\mathbf{C}$ *of tame fields of fixed positive characteristic with divisible value group and fixed residue field* $\mathbb{F}_q$ *(where* $q = p^n$ *for some prime* p *and some* $n \in \mathbb{N}$*) is model complete, complete and decidable. Moreover, it possesses a model of transcendence degree 1 over* $\mathbb{F}_q$ *that admits an elementary embedding in every* $\aleph_1$*-saturated member of* $\mathbf{C}$*.*

b) If "divisible value group" is replaced by "value group elementarily equivalent to $\frac{1}{p^\infty}\mathbb{Z}$*", then* $\mathbf{C}$ *remains elementary, complete and decidable.*

Proof. a): The theory of divisible ordered abelian groups is model complete, complete and decidable, cf. [Ro–Zk] (note that model completeness and decidability are not explicitly stated in the theorems, but follow from their proofs). The same holds trivially for the theory of the finite field $\mathbb{F}_q$ which has only $\mathbb{F}_q$ as a model. Hence, model completeness, completeness and decidability follow readily from Theorem 7.1 and Theorem 1.4. The prime model is constructed as follows: The valued field $(\mathbb{F}_q(t), v_t)$ has value group $\mathbb{Z}$ and residue field $\mathbb{F}_q$. By adjoining suitable roots of t we can build an algebraic extension (F', v_t) with value group $\mathbb{Q}$ and residue field $\mathbb{F}_q$. Now we let (F, v_t) be a maximal immediate algebraic extension of (F', v_t). By Lemma 3.10, it is a tame field. Moreover, it admits $\{t\}$ as a standard valuation transcendence basis over its prime field. Note that $|F| = \aleph_0$. Since $\mathbb{Q}$ is a prime model of the theory of nontrivial divisible ordered abelian groups, Embedding Lemma II (Lemma 5.7) shows that (F, v_t) admits an embedding in every $\aleph_1$-saturated member of $\mathbf{C}$. By the model completeness that we have already proved, this embedding is elementary.

b): The theory of $\frac{1}{p^\infty}\mathbb{Z}$ is clearly complete, and it is decidable (and $\mathbf{C}$ is still elementary) because it can be axiomatized by a recursive set of elementary axioms. Now the proof proceeds as in part a), except that we replace $\mathbb{Q}$ by $\frac{1}{p^\infty}\mathbb{Z}$ and note that the latter admits an elementary embedding in every elementarily equivalent ordered abelian group (again, cf. [Ro–Zk]). □

Note that in the case of b), model completeness can be reinstated by adjoining a constant symbol to the language and adding axioms that state that the value of the element named by this symbol is divisible by no prime but p.

7.2. **Separably defectless and separably tame fields.** We prove part a) of Theorem 1.6:

Assume that $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$. Since vK is cofinal in vL, we know that $(K,v)^c$ is contained in $(L,v)^c$. The compositum $(L.K^c, v)$, taken in the completion $(L,v)^c$, is an immediate extension of (L,v). Thus, $vK^c = vK \prec_\exists vL = vL.K^c$ and $K^c v = Kv \prec_\exists Lv = (L.K^c)v$. Since (K,v) is a henselian separably defectless field, $(K,v)^c$

is henselian by Theorem 32.19 of [W] and defectless by Theorem 5.2 of [K8]. As $(L|K,v)$ is an extension without transcendence defect, the same holds for $(L.K^c|K^c,v)$; indeed, every subextension of $L.K^c|K^c$ of finite transcendence degree is contained in $L'.K^c|K^c$ for some subextension $L'|K$ of finite transcendence degree, and since $(K^c|K,v)$ is immediate, a standard valuation transcendence basis of $(L'|K,v)$ is also a standard valuation transcendence basis of $(L'.K^c|K^c,v)$. By Theorem 1.2, it now follows that

$$(K^c,v) \prec_\exists (L.K^c,v) \; .$$

Let us now take an $|L.K^c|^+$-saturated elementary extension $(K^c|K,v)^*$ of the valued field extension $(K^c|K,v)$. We note that $(K^c,v)^*$ is a subfield of the completion K^{*c} of $(K,v)^*$ since the property of K to be dense in K^c is elementary in the language of valued fields with the predicate $\mathcal{P}$ for the subfield; indeed,

$$\forall x \forall y \exists z : \; \mathcal{P}(z) \, \wedge \, (\, y \neq 0 \, \rightarrow \, v(x-z) > vy \,)$$

expresses this property.

Since $(K^c,v) \prec_\exists (L.K^c,v)$, Proposition 5.1 shows that $(L.K^c,v)$ embeds over K^c in $(K^c,v)^*$. Thus $L.K^c$ can be considered as a subfield of K^{*c}, and so the same holds for the fields L and $L.K^*$. Since $L|K$ is assumed to be separable, it follows that also $L.K^*|K^*$ is separable. Now Theorem 5.9 shows that

$$(K,v)^* \prec_\exists (L.K^*,v) \; .$$

Since $(K,v) \prec (K,v)^*$, we obtain that $(K,v) \prec_\exists (L.K^*,v)$, which yields that $(K,v) \prec_\exists (L,v)$, as asserted. □

We do not know whether the cofinality condition can be dropped.

We can now prove part b) of Theorem 1.6:

Assume that (K,v) is separably tame and that $(L|K,v)$ is a separable extension with $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$. If $\mathrm{char}\, K = 0$, then (K,v) is tame and we have already proved that $(L|K,v)$ satisfies the AKE$^\exists$ Principle. So we assume that $\mathrm{char}\, K = p > 0$. The perfect hull K^{1/p^∞} of K admits a unique extension v of the valuation of K, and with this valuation it is a subextension of the completion of K, according to Lemma 3.20. In particular, $(K^{1/p^\infty}|K,v)$ is immediate. By Lemma 3.21, $(K^{1/p^\infty},v)$ is a tame field. Both K^{1/p^∞} and $L.K^{1/p^\infty}$ are subfields of the perfect hull $(L^{1/p^\infty},v)$ of (L,v), whose value group is the p-divisible hull of vL and whose residue field is the perfect hull of Lv. As $vK = vK^{1/p^\infty}$ is p-divisible and $Kv = Kv^{1/p^\infty}$ is perfect, Lemma 7.2 shows that our side conditions yield that $vK^{1/p^\infty} \prec_\exists v(L.K^{1/p^\infty})$ and $Kv^{1/p^\infty} \prec_\exists (L.K^{1/p^\infty})v$. According to the AKE$^\exists$ Principle for tame fields (Theorem 1.3), this yields that

$$(K^{1/p^\infty},v) \prec_\exists (L.K^{1/p^\infty},v) \; .$$

Now take a $|L.K^{1/p^\infty}|^+$-saturated elementary extension $(K^{1/p^\infty}|K,v)^*$ of $(K^{1/p^\infty}|K,v)$. From this point on, the proof is just an analogue of the proof of the foregoing theorem. □

Related to these results are results of F. Delon [D]. She showed that the **elementary class of algebraically maximal Kaplansky fields of fixed p-degree** is relatively complete. Adding predicates to the language of valued fields which guarantee that every extension is separable, she also obtained relative model completeness. We will discuss the case of separably tame fields of fixed p-degree in a subsequent paper.

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